# Monoids Up to Coherent Homotopy in Two-Level Type Theory

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# Summary

Introduction to Homotopy Type Theory

Introduction to monoids up to coherent homotopy

Two-level Type Theory

(Non-Symmetric) Operads

Construction of $\infty\mathrm{Mon}$

Properties of $\infty\mathrm{Mon}$-algebras

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# Equality in Type Theory

For any type $X$ and any $x, y : X$, we have a type $\mathrm{Id}_X(x, y)$.
So for $p, q : \mathrm{Id}_X(x, y)$ we have the type $\mathrm{Id}_{\mathrm{Id}_X(x,y)}(p, q)$, and so on.

**Question**

How can we interpret those iterated identity types ?

# Homotopy theory

Assume given two continuous maps $f$ and $g$ from $X$ to $Y$.

## Definition

A homotopy from $f$ to $g$ is a continuous map:

$$h : X \times [0, 1] \to Y$$

such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.

## Definition

A homotopy equivalence between two spaces $X$ and $Y$ consists of:

- Continuous maps $f : X \to Y$ and $g : Y \to X$.
- Homotopies from $g \circ f$ to $\mathrm{id}_X$ and $f \circ g$ to $\mathrm{id}_Y$.

Algebraic topology study properties of spaces invariant by homotopy equivalences, mainly homotopy and homology groups.

### Slogan

A lot of different structure model spaces up to homotopy equivalence, for example:

- Simplicial sets.
- $\infty$-groupoids.

# The homotopical interpretation

A type $X$ is a space.
An inhabitant $x : X$ is a point in $X$.
The type $\mathrm{Id}_X(x, y)$ is the space of paths in $X$ from $x$ to $y$.

Then we know how to interpret the whole tower of identity types!

## Important results

- A type together with its iterated identity types has a structure of $\infty$-groupoid in Batanin's sense [van den Berg and Garner, 2010].
- Type theory can be interpreted in simplicial sets [Kapulkin, Lumsdaine and Voevodsky, 2012].
  This interpretation validates the *Univalence Axiom*.
- The Univalence Axiom allows the computation of some homotopy groups [e.g. Brunerie 2016].

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# Introducing coherences

A monoid in Homotopy Type Theory should be given by:

- $X : \mathrm{Type}$
- $\_ \times \_ : X \to X \to X$
- $1 : X$

together with for all $a, b, c : X$ a path:

$$\begin{tikzcd} a \times (b \times c) \arrow[r, dash] & (a \times b) \times c \end{tikzcd}$$

and for all $a : X$ two paths:

$$\begin{tikzcd} a \times 1 \arrow[r, dash] & a \end{tikzcd}$$

$$\begin{tikzcd} 1 \times a \arrow[r, dash] & a \end{tikzcd}$$

But this is not enough!

For example, for all $a, b, c, d : X$, one should have a filling of:

$$\begin{tikzcd}
& a \times ((b \times c) \times d) \arrow[rr, dash] \arrow[dl, dash] && (a \times (b \times c)) \times d \arrow[dr, dash] & \\
a \times (b \times (c \times d)) \arrow[drr, dash] &&&& ((a \times b) \times c) \times d \arrow[dll, dash] \\
&& (a \times b) \times (c \times d) &&
\end{tikzcd}$$

**and so on...**

# Classical theory

Such monoids are well-known in algebraic topology, and are called *monoids up to coherent homotopy*, or $\infty$-*monoids*.

### Theorems [May, Boardmann and Vogt, 60s]

1. If $X$ is an $\infty$-monoid and $X \simeq Y$, then $Y$ is an $\infty$-monoid.
2. If $X$ is an $\infty$-monoid, then there exists a topological monoid $Y$ with $X \simeq Y$.
3. Any loop space is an $\infty$-monoid with concatenation of paths as multiplication. It is said *group-like*.
4. Any group-like $\infty$-monoid is equivalent to a loop space.

We define $\infty$-monoids in Agda and prove 1 and 3.

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# A type theory with two equalities

It is well-known that it is hard to handle infinite towers of coherences in plain Homotopy Type Theory.

Following Voevodsky's idea of a *Homotopy Type System*, we implement in Agda a type theory with two equalities:

- A strict equality $\_ \equiv \_$ which obeys Axiom K and function extensionnality, interpreted as the usual mathematical equality.
- A homotopical equality $\_ \rightsquigarrow \_$ interpreted as the type of paths between two points. It is intended to obey a form of univalence.

# Definition of our extension of Agda

We use the default equality of Agda as a strict equality. It obeys Axiom K, and we postulate function extensionnality.

**Assumption**

We postulate a predicate:

$$\mathrm{isFibrant} : \mathrm{Type} \to \mathrm{Type}$$

stable by Σ, Π, $\top$ and isomorphisms.

**Assumption**

We postulate a type $\mathbb{I}$ called the interval together with $0, 1 : \mathbb{I}$ such that:

- If $X$ is fibrant and $C : (\mathbb{I} \to X) \to \mathrm{Type}$ is a family of fibrant types, then given $d : (x : X) \to C(\lambda i.x)$ we have:

$$J(d) : (p : \mathbb{I} \to X) \to C(p)$$

Moreover we assume that if $P : \mathbb{I} \to \mathrm{Type}$ is a family of fibrant types, the type of $f : (i : \mathbb{I}) \to P(i)$ with fixed endpoints is fibrant.

### Definition

Given $X : \mathrm{Type}$ and $x, y : X$, we can define the type of paths from $x$ to $y$ as:

$$x \rightsquigarrow y \quad := \quad \Sigma(f : \mathbb{I} \to X).\ f(0) \equiv x \wedge f(1) \equiv y$$

We have path elimination only into fibrant types.

Now we can use all the usual homotopical definitions, and they behave as expected in the universe of fibrant types.

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# Intuitions about operads

Intuitively operads correspond to certain well-behaved linear algebraic theory, for example the theory of monoids.

An operad in $\mathrm{Set}$ is a family $\mathcal{P} : \mathbb{N} \to \mathrm{Set}$ with some structure, where $\mathcal{P}(n)$ is interpreted as the set of $n$-ary operations derived from the algebraic theory.

### Key example

The operad $\mathrm{Mon}$ defined by:

$$\mathrm{Mon}(n) \quad := \quad \top$$

corresponds to monoids.

Operads can be defined in any monoidal category.

# Definition of operads in the category of types.

**Definition**

An operad consists of:

- $\mathcal{P} : \mathbb{N} \to \mathrm{Type}$
- $\mathrm{id}_{\mathcal{P}} : \mathcal{P}(1)$
- $\gamma_{\mathcal{P}} : \mathcal{P}(n) \to \mathcal{P}(k_1) \to \cdots \to \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \cdots + k_n)$

obeying axioms suggested by the interpretation of $\mathcal{P}(n)$ as $n$-ary operations.

**Definition**

A morphism of operads $\alpha : \mathcal{P} \to \mathcal{Q}$ is a map:

$$\alpha : (n : \mathbb{N}) \to \mathcal{P}(n) \to \mathcal{Q}(n)$$

commuting with $\mathrm{id}$ and $\gamma$.

# Algebras for an operad

### Definition

For $X : \mathrm{Type}$, we have an operad $\mathcal{E}nd_X$ with:

$$\mathcal{E}nd_X(n) \quad := \quad X^n \to X$$

### Definition

We says that $X : \mathrm{Type}$ is a $\mathcal{P}$-algebra if we are given a morphism of operad:

$$\epsilon_X : \mathcal{P} \to \mathcal{E}nd_X$$

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# Reformulating our goal

We construct an operad $\infty\mathrm{Mon}$ such that:

1. For all $n : \mathbb{N}$, the type $\infty\mathrm{Mon}(n)$ is contractible.
2. Assume $X, Y$ two fibrant types such that $X \simeq Y$. If $X$ is an $\infty\mathrm{Mon}$-algebra, so is $Y$.
3. For any $X$ fibrant and $x : X$, the type $x \rightsquigarrow x$ is an $\infty\mathrm{Mon}$-algebra.

# Labelled Trees

### Definition

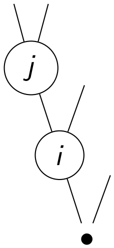
We define the type of trees inductively:

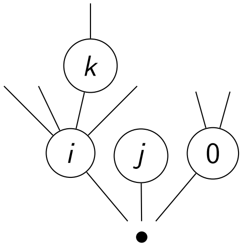
- $\mathrm{leaf} : \mathrm{Tree}$
- $\mathrm{cons} : (n : \mathbb{N}) \to (\mathrm{Fin}(n) \to \mathrm{Tree}) \to \mathrm{Tree}$

### Definition

We define a labelled tree as a tree together with a labelling of its internal vertices by elements of $\mathbb{I}$.

Graphical representation of labelled trees:



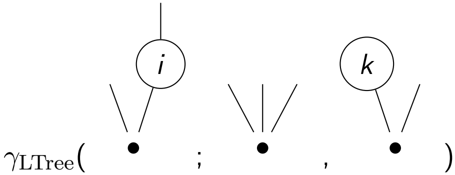


We denote $\mathrm{LTree}(n)$ the type of labelled trees with arity $n$.

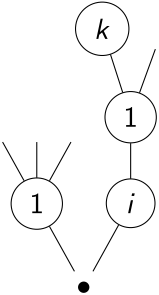
**Lemma**

There is an operad structure on $\mathrm{LTree}$.
Composition is defined as the grafting of tree with 1 added on the new internal vertices.

For example

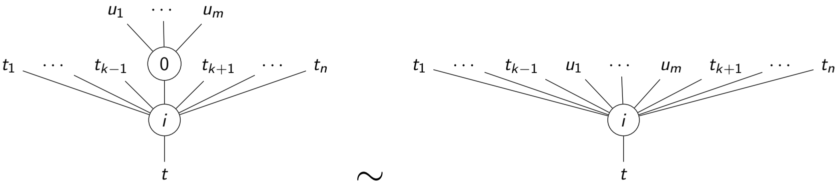


is



# Definition of $\infty\mathrm{Mon}$

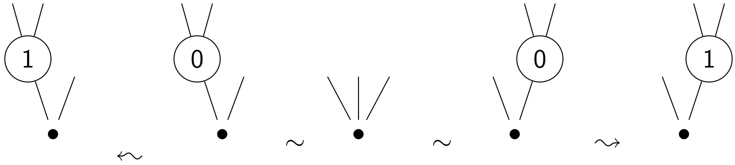
We can define $\infty\mathrm{Mon}$ as the *strict* quotient of $\mathrm{LTree}$ by a relation $\sim$ including:



And some other relations for unary vertices.

Let $X$ be a fibrant $\infty\mathrm{Mon}$-algebra.

The image of  gives a binary operation $X \to X \to X$. We have:



so this operation is associative up to homotopy.

More generally:

**Proposition**

For all $n : \mathbb{N}$, the type $\infty\mathrm{Mon}(n)$ is contractible.

So we have all the coherences we want.

# Key property of $\infty\mathrm{Mon}$

## Definition

A morphism of operad $\alpha : \mathcal{P} \to \mathcal{Q}$ is said strongly contractible if for all $k, n : \mathbb{N}$ we can solve:

$$\begin{tikzcd}
\partial\mathbb{I}^k \arrow[r] \arrow[d, "i"] & \mathcal{P}(n) \arrow[d, "\alpha(n)"] \\
\mathbb{I}^k \arrow[r] \arrow[ur, dashed] & \mathcal{Q}(n)
\end{tikzcd}$$

## Theorem

Let $\beta : \mathcal{P} \to \infty\mathrm{Mon}$ be a strongly contractible morphism of operads, then it has a section **which is a morphism of operads**.

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# Invariance under equivalence

**Definition**

A map is called a trivial fibration if its fibre are fibrant and contractible.

**Definition**

An operad $\mathcal{P}$ is called cofibrant if for any trivial fibration of operad $\alpha : \mathcal{Q}_1 \to \mathcal{Q}_2$ with fibrant base we can solve:

$$\begin{tikzcd}
 & \mathcal{Q}_1 \arrow[d, "\alpha"] \\
\mathcal{P} \arrow[ur, dashed] \arrow[r] & \mathcal{Q}_2
\end{tikzcd}$$

### Proposition

If $\mathcal{P}$ is cofibrant and $X, Y$ are fibrant types such that $X \simeq Y$, then if $X$ is a $\mathcal{P}$-algebra so is $Y$.

### Proposition

$\infty\mathrm{Mon}$ is cofibrant.

# Loop spaces are $\infty\mathrm{Mon}$-algebras

**Lemma**

Assume given $X$ a type, then there exists an operad $\mathcal{P}ath_X$ with $\mathcal{P}ath_X(n)$ defined as:

$$\Sigma(\phi : (x_0, ..., x_n : X) \to x_0 \rightsquigarrow x_1 \to \cdots \to x_{n-1} \rightsquigarrow x_n \to x_0 \rightsquigarrow x_n).$$

$$(x : X) \to \phi(\mathrm{refl}_x, \cdots, \mathrm{refl}_x) \equiv \mathrm{refl}_x$$

For any $x : X$ we have a morphism of operad $\mathcal{P}ath_X \to \mathcal{E}nd_{x \rightsquigarrow x}$.

### Lemma

For any fibrant $X$ and $k, n : \mathbb{N}$ we can solve:

$$\begin{tikzcd}
\partial \mathbb{I}^k \arrow[r] \arrow[d, "i"] & \mathcal{P}ath_X(n) \\
\mathbb{I}^k \arrow[ur, dashed] &
\end{tikzcd}$$

So there is a morphism of operad $\infty\mathrm{Mon} \to \mathcal{P}ath_X$.