Differential Geometry in Synthetic Algebraic Geometry

Hugo Moeneclaey
j.w.w Felix Cherubini, Matthias Hutzler and David Wärn

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Overview

Goal

Import differential geometry tools to synthetic algebraic geometry.

Draft

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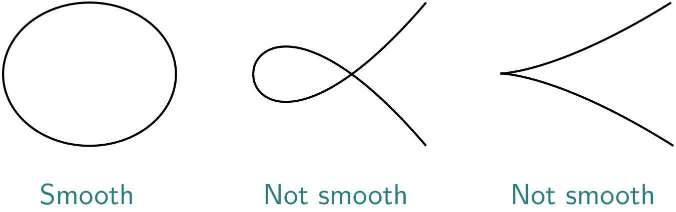

Outline

Synthetic algebraic geometry

Smoothness for arbitrary types

Smoothness for affine schemes

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R is a set.

Affine schemes

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then:

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Definition

A type X is an affine scheme if there is an f.p. algebra A such that:

$$X = Spec(A)$$

Axiom 2: Duality

For any f.p. algebra A the map:

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Then:

- $Spec : \{f.p.\ algebras\} \simeq \{Affine\ schemes\}$
- All maps between affine schemes are polynomials.

Axiom 3: Zariski local choice

Affine schemes enjoys a weakening of the axiom of choice.

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Lemma

Let P be a closed proposition, TFAE:

(1) There exist $r_1, \ldots, r_n : R$ nilpotent such that:

$$P \leftrightarrow (r_1 = 0 \wedge \ldots \wedge r_n = 0)$$

(2) $\neg\neg P$.

Such a proposition is called closed dense.

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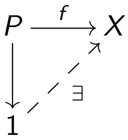

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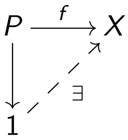

What does this has to do with smoothness?

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The affine scheme:

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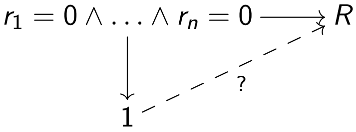

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We have to merely find a lift in:

$$\begin{array}{ccc} r_1 = 0 \wedge \ldots \wedge r_n = 0 & \longrightarrow & R \\ \downarrow & \overset{?}{\nearrow} & \\ 1 & & \end{array}$$

By duality it is enough to merely find a lift in:

$$\begin{array}{ccc} R/(r_1, \ldots, r_n) & \longleftarrow & R[X] \\ \uparrow & \swarrow & \\ R & & \end{array}$$

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The affine scheme:

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gives $r, r' : (\epsilon^2)$ such that $(\epsilon + r)(\epsilon + r') = 0$, so that $\epsilon^2 = 0$.

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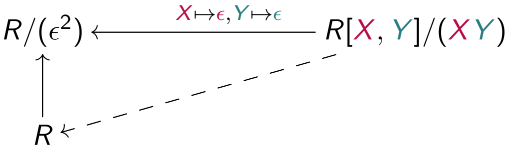

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Then:

$$\{x : R \mid x^2 = 0\} = \{x : R \mid x^3 = 0\}$$

which by duality implies:

$$R[X]/(X^2) = R[X]/(X^3)$$

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For any map $f : X \to Y$ and $p : X$ we have the differential:

$$df_p : T_p(X) \to T_{f(p)}(Y)$$

Maps with smooth fibers

Proposition

Let $f : X \to Y$ be a map between affine schemes with X smooth. TFAE:

- For all $p : X$ the differential df_p is surjective.
- The fibers of f are smooth.

Tangent spaces of smooth affine schemes

A module M is:

- Finite free if there is $k : \mathbb{N}$ such that $M = R^k$.
- Finitely copresented if it is the kernel of a map $R^m \to R^n$.

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General idea:

1. Tangent spaces of affine schemes are finitely copresented.
2. Tangent space of smooth affine schemes are smooth.
3. Smooth finitely copresented modules are finite free.

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- $X^{\mathbb{D}}$ is an affine scheme as a dependent sum of affine schemes.
- $X^{\mathbb{D}}$ is smooth as X is smooth and $\mathbb{D}$ has choice.
- We need to check its differentials are surjective.

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But we can do this as X is smooth.

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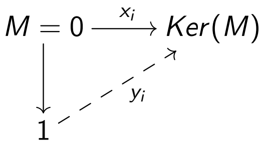
Assume $\neg\neg(M = 0)$. Take (x_i) a basis of R^m, we have lifts in:



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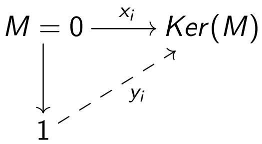

But:

- If $M = 0$ then (y_i) is equal to (x_i) so it is a basis of R^m.
- We have $\neg\neg(M = 0)$.
- Being a basis is $\neg\neg$-stable.

So (y_i) is a basis of R^m and $Ker(M) = R^m$ so $M = 0$.

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By induction on m. Apply the previous lemma.

- If $M = 0$, then $Ker(M) = R^m$ and it is finite free.
- If $M \neq 0$, then M has an invertible coefficient. By Gaussian elimination we get a linear map $N : R^{m-1} \to R^{n-1}$ with the same kernel.

Conclusion

Today:

- Showcased a couple of synthetic proofs.
- Gave some nice properties of smoothness for affine schemes.

In the notes:

- Justify smoothness through its connections with étaleness.
- Prove smoothness for general types is well-behaved.
- Give an explicit Zariski local description of smooth schemes.
- And much more!

Appendix: Explicit Zariski local description

Definition

A standard smooth scheme is an affine scheme of the form:

$$Spec\Big(\big(R[X_1, \ldots, X_n, Y_1, \ldots, Y_k]/(P_1, \ldots, P_n)\big)_G\Big)$$

where $Jac(P_1, \ldots, P_n) \mid G$.

Theorem

Let X be a scheme, TFAE:

- X is smooth.
- X has a finite open cover by standard smooth schemes.

Appendix: Smoothness is well behaved

Lemma

Open propositions are smooth.

Lemma

Smooth types are closed by Σ.

Lemma

If D has choice and X is smooth, then X^D is smooth.

Lemma

The image of a smooth type by any map is smooth.

Lemma

A type X is smooth if and only is $||X||_0$ is smooth.