In Memory of Tim Lichtnau: Deligne-Mumford Stacks in Synthetic Algebraic Geometry

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after Tim Lichtnau's master thesis notes

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Tim Lichtnau

Tim died on the 26th of December 2024.

In this talk I will present work from his unfinished master thesis.

Outline

From affine schemes to Deligne-Mumford stacks

Synthetic Deligne-Mumford stacks

Algebraic spaces that are not schemes

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Affine schemes

Duality axiom

We have an equivalence

$$R\text{-Alg}_{fp} \simeq \{\text{Affine schemes}\}.$$

For affine schemes, algebra and geometry match up perfectly.

But not everything is affine

Assume $n > 0$.

Definition

We define the projective space $\mathbb{P}^n$ by

$$\mathbb{P}^n = \{\text{Lines in } R^{n+1} \text{ going through } 0\}.$$

Any map from $\mathbb{P}^n$ to R is constant, so $\mathbb{P}^n$ cannot be affine.

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A scheme is a set that has a finite open cover by affine pieces.

$\mathbb{P}^n$ is a scheme.

But not everything is a scheme

Definition

A cubic surface is a scheme X such that there exists $P : R[X_0, \cdots, X_3]$ homogeneous of degree 3 with

$$X = \{x : \mathbb{P}^3 \mid P(x) = 0\}.$$

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Remark

The type of smooth cubic surfaces is nice:

- We have a surjection $\mathbb{P}^{19} \to \{\text{Cubic surfaces}\}$.
- Being smooth is an open condition.

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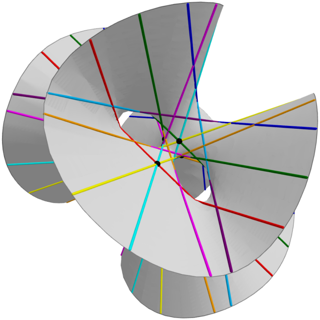
- Being smooth is an open condition.

But it cannot be a scheme, it is not even a set!

Automorphisms of cubic surfaces

Theorem (over k an algebraically closed field)

Any smooth cubic surface contains precisely 27 lines.



This gives a tight control on automorphisms of these surfaces.

Deligne-Mumford stacks

The notion of Deligne-Mumford stack models this situation.

Intuition

A Deligne-Mumford stack is a 1-type that is:

- an étale sheaf.
- étale-locally affine.

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Remark

[Simpson 96] gave an extension to n-types.

Tim worked on a synthetic version of this extended version.

Why do it synthetically?

Traditional approach

Use sheaves of n-groupoids (aka n-stacks) on the étale site.

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Use sheaves of n-groupoids (aka n-stacks) on the étale site.

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Synthetic approach

- Sheaves of n-groupoids on the Zariski site are just n-types.
- Being an étale sheaf is an accessible lex modality.

$\Rightarrow$ We expect a more direct and elegant development.

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Formally étale types

Definition

A type X is formally étale if given any $\epsilon : R$ nilpotent, the map:

$$X \to X^{\epsilon=0}$$

is an equivalence.

Intuition

- ϵ nilpotent means ϵ infinitesimal.
- X formally étale means X discrete.

Étale sheaves

Definition

A type X is an étale sheaf if for all S affine, formally étale and non-empty, we have that

$$X \to X^{\|S\|}$$

is an equivalence.

If our goal is an étale sheaf, we can assume e.g. $X^2 + 1$ has a root[1].

[1] If 2 is invertible in R.

Deligne-Mumford stacks

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Definition

DM is the smallest class of étale sheaves such that:

- S affine $\Rightarrow S \in \mathrm{DM}$.
- (S affine, $p : S \to X$ étale cover whose fibers are in DM) $\Rightarrow X \in \mathrm{DM}$.

General properties of Deligne-Mumford stacks

Proposition

- DM is stable under Σ and identity types.
- DM is stable under étale quotients[2].

[2]Meaning given Y an étale sheaf with $X \in \mathrm{DM}$ and $p : X \to Y$ an étale cover with fibers in DM, we have that $Y \in \mathrm{DM}$.

General properties of Deligne-Mumford stacks

Proposition

- DM is stable under Σ and identity types.
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Proposition

The type of Deligne-Mumford stacks is an étale sheaf.

Traditionally called étale descent for Deligne-Mumford stacks.

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Algebraic spaces

Definition

An algebraic space is a Deligne-Mumford stack which is a set[3].

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- A scheme is a set that is Zariski-locally affine.
- An algebraic space is a set that is étale-locally affine.

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- A scheme is a set that is Zariski-locally affine.
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We want non-trivial examples of algebraic spaces.

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Algebraic spaces that are not schemes

Tim proved the following chain of strict inclusions.

$$\{\text{Schemes}\} \subsetneq \left\{ \begin{array}{c} \text{Locally separated} \\ \text{algebraic spaces} \end{array} \right\} \subsetneq \{\text{Algebraic spaces}\}$$

An algebraic space that is not locally separated

Assume 2 invertible in R.

Lemma

For $x, y : R$, we define

$$(x \sim y) = (x = y) \vee \big((x \neq 0) \wedge (y \neq 0) \wedge (x = -y)\big).$$

Lemma

$R/\sim$ is an algebraic space that is not locally separated.

A locally separated algebraic space that is not a scheme

Assume 2 invertible in R.

Definition

We define

$$H = \{r : R^\times, x : R, y : R/x \mid y^2 = r \bmod x\}.$$

Proposition

- H is a locally separated algebraic space.
- H is not a scheme.

The future of Tim's work

- A synthetic proof of smooth cubic surfaces forming a Deligne-Mumford 1-stack has never been closer[4].
- I am working on generalising Tim's definition to abstract sheaf models.
- Tim's work should prove foundational for étale synthetic algebraic geometry.

[4] Ongoing work with Felix Cherubini, Thierry Coquand, Fabian Endres, Jonas Höfer and Marc Nieper-Wißkirchen