Cohomology in Synthetic Stone Duality

Hugo Moeneclaey

j.w.w. Felix Cherubini, Thierry Coquand and Freek Geerligs

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Overview

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Cohomology in HoTT

Given $n : \mathbb{N},\ X : \text{Type},\ A : X \to \text{Ab}$, we define a group $H^n(X, A)$.

$H^n(X, A)$ is the n-th cohomology group of X with coefficient A.

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Our previous work [CCGM24]

- Showed SSD is suitable for synthetic topological study of Stone and compact Hausdorff spaces.
- Proved $H^1(X, \mathbb{Z})$ is well-behaved for X : CHaus .

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1. Introduce SSD, Stone spaces and compact Hausdorff spaces.
2. Introduce the cohomology groups $H^n(X, A)$.
3. Introduce overtly discrete types and Barton-Commelin axioms:

$$\Pi_{x:X} I(x) \text{ is well-behaved for } X : \mathsf{CHaus} \text{ and } I : X \to \mathsf{ODisc}.$$

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Plan

1. Introduce SSD, Stone spaces and compact Hausdorff spaces.
2. Introduce the cohomology groups $H^n(X, A)$.
3. Introduce overtly discrete types and Barton-Commelin axioms:
   $\Pi_{x:X} I(x)$ is well-behaved for X : CHaus and $I : X \to \mathrm{ODisc}$.
4. Explain our main results:
   $H^n(X, A)$ is well-behaved for X : CHaus and $A : X \to \mathrm{Ab}_{\mathrm{ODisc}}$.

An abelian group is overtly discrete iff it is countably presented.

Outline

SSD, Stone spaces and compact Hausdorff spaces

Intoduction to cohomology in HoTT

Overtly discrete types and Barton-Commelin axioms

Cohomology of Stone and compact Hausdorff spaces

Stone spaces

Definition

A type X is a Stone space if it is a sequential limit of finite types.

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Example 1: Cantor space

The type $2^{\mathbb{N}}$ is a Stone space.

Indeed $2^{\mathbb{N}} = \lim_{i:\mathbb{N}} 2^i$.

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Example 2: Compactification of $\mathbb{N}$

The type:

$$\mathbb{N}_\infty = \{\alpha : 2^{\mathbb{N}} \mid \alpha \text{ hits 1 at most once}\}$$

is a Stone space.

Indeed $\mathbb{N}_\infty$ is the limit of:

$$\mathsf{Fin}(1) \xleftarrow{-1} \mathsf{Fin}(2) \xleftarrow{-1} \mathsf{Fin}(3) \xleftarrow{-1} \mathsf{Fin}(4) \xleftarrow{-1} \cdots$$

$$\begin{tikzcd}
\{\} & \{0\} \arrow[l] & \{00\} \arrow[l] & \{000\} \arrow[l] & \cdots \arrow[l] \\
 & \{1\} \arrow[ul] & \{01\} \arrow[ul] & \{001\} \arrow[ul] & \ddots \arrow[ul] \\
 & & \{10\} \arrow[ul] & \{010\} \arrow[ul] & \ddots \arrow[ul] \\
 & & & \{100\} \arrow[ul] & \ddots \arrow[ul] \\
 & & & & \ddots \arrow[ul]
\end{tikzcd}$$

Synthetic Stone duality

Axiom 1a: Scott continuity

If $(S_k)_{k:\mathbb{N}}$ is a tower of finite types, then any map in $(\lim_k S_k) \to 2$ merely factors through an S_k.

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Axiom 1b: Markov's principle

If $(D_k)_{k:\mathbb{N}}$ are decidable propositions, $\neg(\forall_{k:\mathbb{N}} D_k) \to \exists(k : \mathbb{N}). \neg D_k$.

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Axiom 2: Weak König's lemma

If $(S_k)_{k:\mathbb{N}}$ is a tower of inhabited finite types, then $\|\lim_k S_k\|$.

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Axiom 4: Dependent choice

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Axiom 3: Local choice

Assume given S : Stone and $Y : S \to \text{Type}$ such that $\Pi_{s:S}\|Y(s)\|$. Then there exists T : Stone and $p : T \twoheadrightarrow S$ such that $\Pi_{t:T} Y(p(t))$.

Compact Hausdorff spaces

Stone spaces are not stable under quotients.

Definition

A set X is a compact Hausdorff space if:

- Its identity types are Stone spaces.
- There exists S : Stone and $S \twoheadrightarrow X$.

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Example: The unit interval

The type $\mathbb{I} = [0, 1]$ is a compact Hausdorff space.

Indeed $\mathbb{I}$ is a quotient of $2^{\mathbb{N}}$.

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SSD, Stone spaces and compact Hausdorff spaces

Intoduction to cohomology in HoTT

Overtly discrete types and Barton-Commelin axioms

Cohomology of Stone and compact Hausdorff spaces

Delooping abelian groups

Fix A an abelian group. We define $K(A, 0) = A$.

Proposition

Given $n > 0$, there is a unique pointed type $K(A, n)$ such that:

- $K(A, n)$ is $(n\text{-}1)$-connected and n-truncated.
- $\Omega^n K(A, n) = A$.

$K(A, n)$ is called the *n-th delooping of A*.

Cohomology groups

Definition: Cohomology

Given $n : \mathbb{N}$, X : Type and $A : X \to \mathrm{Ab}$, we define

$$H^n(X, A) = \|\Pi_{x:X} K(A_x, n)\|_0.$$

Cohomology groups

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Given $n : \mathbb{N}$, X : Type and $A : X \to \mathrm{Ab}$, we define

$$H^n(X, A) = \|\Pi_{x:X}\, K(A_x, n)\|_0.$$

Remark: Why cohomology?

- If $H^n(X, A) = 0$ then we can use some choice on X.
- There exists many tools to compute cohomology.

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Definition

A type is overtly discrete if it is a sequential colimit of finite types.

An abelian group is overtly discrete iff it is countably presented.

Tychonov and its dual

We prove Barton-Commelin's condensed type theory axioms.

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Lemma: Tychonov

If I : ODisc and $X : I \to$ CHaus, then $\Pi_{i:I} X_i$ is compact Hausdorff.

Tychonov and its dual

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Lemma: Tychonov

If I : ODisc and $X : I \to$ CHaus, then $\Pi_{i:I} X_i$ is compact Hausdorff.

Proposition: Tychonov's dual

If X : CHaus and $I : X \to$ ODisc, then $\Pi_{x:X} I_x$ is overtly discrete.

This is encouraging. We have better!

Scott continuity

Definition

We have a category $\mathcal{C}$ where:

$$\begin{aligned} \mathrm{Ob}_{\mathcal{C}} &= \Sigma(X : \mathsf{CHaus}).\, X \to \mathsf{ODisc} \\ \mathrm{Hom}_{\mathcal{C}}((X, I), (Y, J)) &= \Sigma(f : Y \to X).\, \Pi_{y:Y}\, I_{f(x)} \to J_x \end{aligned}$$

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Theorem: Generalized Scott continuity

The functor $\Pi : \mathcal{C} \to \mathrm{ODisc}$ commutes with sequential colimits.

Outline

Čech cohomology

Definition

A Čech cover consists of X : CHaus and S : Stone with $p : S \twoheadrightarrow X$.

Definition

Given a Čech cover $p : S \twoheadrightarrow X$ and $A : X \to \mathrm{Ab}_{cp}$, we define $\check{H}^n(X, S, A)$ as the n-th cohomology group of

$$\Pi_{x:X} A_x^{S_x} \to \Pi_{x:X} A_x^{S_x^2} \to \Pi_{x:X} A_x^{S_x^3} \to \cdots .$$

$\check{H}^n(X, S, A)$ is called the n-th Čech cohomology group of X with coefficient in A.

Main results

Theorem: Cohomology vanishing for Stone spaces

Given $n > 0$, S : Stone and $A : S \to \mathrm{Ab}_{cp}$, we have that

$$H^n(S, A) = 0.$$

Theorem: Čech and regular cohomology agree on CHaus

Given a Čech cover $p : S \twoheadrightarrow X$ and $A : X \to \mathrm{Ab}_{cp}$, we have that

$$H^n(X, A) = \check{H}^n(X, S, A).$$

Applications

Lemma: Cohomology of the interval

For $A : \mathrm{Ab}_{cp}$, we have that

$$H^n(\mathbb{I}, A) = \begin{cases} A & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma: Cohomology of the spheres

For $\mathbb{S}^k = \{x_0, \ldots, x_k : \mathbb{R} \mid \Sigma_i x_i^2 = 1\}$ and $A : \mathrm{Ab}_{cp}$, we have that

$$H^n(\mathbb{S}^k, A) = \begin{cases} A & \text{if } n = 0 \text{ or } n = k \\ 0 & \text{otherwise.} \end{cases}$$

This extends to all countable topological CW complex.

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Axiom 1b: Markov's principle

If $(D_k)_{k:\mathbb{N}}$ are decidable propositions, $\neg(\forall_{k:\mathbb{N}} D_k) \to \exists(k : \mathbb{N}). \neg D_k$.

Axiom 2: Weak König's lemma

If $(S_k)_{k:\mathbb{N}}$ is a tower of inhabited finite types, then $\|\lim_k S_k\|$.

Axiom 4: Dependent choice

If $(X_k)_{k:\mathbb{N}}$ is a tower with surjective maps, then $\lim_k X_k \twoheadrightarrow X_0$.

Axiom 3: Local choice

Assume given S : Stone and $Y : S \to \mathrm{Type}$ such that $\Pi_{s:S} \|Y(s)\|$. Then there exists T : Stone and $p : T \twoheadrightarrow S$ such that $\Pi_{t:T} Y(p(t))$.