Châtelet's Theorem in Synthetic Algebraic Geometry

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Overview

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Theorem [Châtelet 1944]

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Goal

Give a synthetic proof of Châtelet's theorem.

Overview

Today

1. Introduce synthetic algebraic geometry.
2. Define étale sheaves. Being an étale sheaf is a lex modality.
3. Use étale sheafification to define Severi-Brauer varieties.
4. Sketch the proof of Châtelet's theorem.
   Essentially lex modality reasoning + some linear algebra.

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4. Sketch the proof of Châtelet's theorem.
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Main point of this talk

Working with étale sheaves in synthetic algebraic geometry is convenient and helpful.

Outline

Introduction to synthetic algebraic geometry

Étale sheaves

Severi-Brauer varieties and Châtelet's theorem

What is synthetic algebraic geometry?

It consists of HoTT plus 3 axioms:

Axiom 1

There is a local ring R.

R is assumed to be a set.

Affine schemes

For A a finitely presented algebra, we define:

$$Spec(A) = Hom_{R\text{-}Alg}(A, R)$$

Example

If:

$$A = R[X]/P$$

then:

$$Spec(A) = \{x : R \mid P(x) = 0\}$$

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Definition

A type X is an affine scheme if there is an f.p. algebra A such that:

$$X = Spec(A)$$

Axiom 2: Duality

For any f.p. algebra A the map:

$$A \to R^{Spec(A)}$$

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Then:

- $Spec : \{f.p.\ algebras\} \simeq \{Affine\ schemes\}$
- All maps between affine schemes are polynomials.

Axiom 3: Zariski local choice

Affine schemes enjoys a weakening of the axiom of choice.

Schemes

Defintion

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Example

The projective space:

$$\mathbb{P}^n = \{Lines\ in\ R^{n+1}\ going\ through\ the\ origin\}$$

is a scheme.

Châtelet's theorem in traditional algebraic geometry

Definition (Traditional algebraic geometry, k a field)

A Severi-Brauer variety over k is a k-scheme X such that X is a projective space over the separable closure of k.

Theorem [Châtelet 1944]

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Challenge

How to define synthetic Severi-Brauer varieties?

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A type X is an étale sheaf if for any monic unramifiable $P : R[X]$, we have a unique filler:

$$\begin{tikzcd} \exists (x : R).\, P(x) = 0 \arrow[r] \arrow[d] & X \\ 1 \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

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Remark

By [Wraith 79], this should correspond to traditional étale sheaves.

Being an étale sheaf is a lex modality.
So we have an étale sheafification [Rijke, Shulman, Spitters 2017].

Remark

HoTT can be interpreted inside étale sheaves:

- The universe U is interpreted as the type U_{Et} of étale sheaves.
- Σ, Π and identity types are interpreted as themselves.
- Truncation $\|\_\|$ is interpreted as the étale sheafification of the truncation, denoted $\|\_\|_{Et}$.

Required properties of étale sheaves

Lemma

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Let M be a module that is an étale sheaf.
The proposition "M is finite free" is an étale sheaf.

Equivalently, the type of finite free module is an étale sheaf.

Remark

Traditionally phrased as étale descent for finite free modules.

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Severi-Brauer varieties

We fix a natural number n.

Definition

A Severi-Brauer variety is an étale sheaf X such that $\|X = \mathbb{P}^n\|_{Et}$.

We write:

$$SB_n = \{X : U_{Et} \mid \|X = \mathbb{P}^n\|_{Et}\}$$

Example: Conics

Assume $2 \neq 0$.

Example

For all $a, b : R$ invertible:

$$C(a, b) = \{[x : y : z] : \mathbb{P}^2 \mid ax^2 + by^2 = z^2\}$$

is a Severi-Brauer variety.

Severi-Brauer varieties form a delooping

Theorem [Cherubini, Coquand, Hutzler, Wärn 2024]

$$Aut(\mathbb{P}^n) = PGL_{n+1}(R)$$

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$$Aut(\mathbb{P}^n) = PGL_{n+1}(R)$$

Therefore:

$$SB_n = \{X : U_{Et} \mid \|X = \mathbb{P}^n\|_{Et}\}$$

is the interpretation inside étale sheaves of:

$$\{X : U \mid \|X = \mathbb{P}^n\|\} = B(Aut(\mathbb{P}^n)) = B(PGL_{n+1}(R))$$

Azumaya algebras form a delooping

Lemma

$$Aut_{R\text{-}Alg}(M_{n+1}(R)) = PGL_{n+1}(R)$$

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So we define the type of Azumaya algebras:

$$AZ_n = \{A : R\text{-}Alg_{Et} \mid \|A = M_{n+1}(R)\|_{Et}\}$$

It is also the interpretation of $B(PGL_{n+1}(R))$ inside étale sheaves.

Severi-Brauer varieties from Azumaya algebras

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Lemma

The pointed map:

$$\begin{aligned} LI &: AZ_n \to SB_n \\ LI(A) &= \{I \text{ left ideals in } A \mid I \text{ free of rank } n+1\} \end{aligned}$$

is an equivalence.

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Proof sketch

LI induces a map:

$$\alpha : PGL_{n+1}(R) = \Omega AZ_n \xrightarrow{\Omega LI} \Omega SB_n = PGL_{n+1}(R)$$

By lex modality reasoning, we just need that α is the identity.
Check this by direct computation.

Lemma

Given $A : AZ_n$, if we have $I : LI(A)$ then $A = End_R(I)$.

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Then we do some linear algebra.

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Corollary

Given $A : AZ_n$, if we have $\|LI(A)\|$ then $\|A = M_{n+1}(R)\|$.

Châtelet's theorem in synthetic algebraic geometry

Theorem

For all $X : SB_n$, we have that $\|X\|$ implies $\|X = \mathbb{P}^n\|$.

Proof

$$\begin{array}{rcl} & \|X\| & \\ \Rightarrow & \|LI(A)\| & (\textit{write } A = LI^{-1}(X)) \\ \Rightarrow & \|A = M_{n+1}(R)\| & (\textit{previous Corollary}) \\ \Rightarrow & \|X = LI(M_{n+1}(R))\| & (\textit{apply } LI) \\ \Rightarrow & \|X = \mathbb{P}^n\| & (LI \textit{ pointed}) \end{array}$$