Differential Geometry of Synthetic Schemes

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Abstract

Synthetic algebraic geometry uses homotopy type theory extended with three axioms to develop algebraic geometry internal to a higher version of the Zariski topos. In this article we make no essential use of the higher structure and use homotopy type theory only for convenience. We define étale, smooth and unramified maps between schemes in synthetic algebraic geometry using a new synthetic definition. We give the usual characterizations of these classes of maps in terms of injectivity, surjectivity and bijectivity of differentials. We also show that the tangent spaces of smooth schemes are finite free modules. Finally, we show that unramified, étale and smooth schemes can be understood very concretely via the expected local algebraic description.

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Introduction

In mathematics, it is common practice to assume a fixed set theory, usually with the axiom of choice, as a common basis. While it is a great advantage to work in one common language and share a lot of the basic constructions, the dual approach of adapting the "base language" to particular mathematical domains is sometimes more concise, provides a new perspective and encourages new proof techniques which would be hard to find otherwise. We use the word "synthetic" to indicate that the latter approach is used, as it was used by Lawvere, when he described a program to develop mathematics inside of certain categories [Law79].

Just using category theory is not the same as reasoning synthetically – for the latter the goal is usually to derive results exclusively in one system, as Lawvere did with differential geometry in his work. The distinction with just using an abstraction like categories is important, since the translation from the synthetic language and back can become quite cumbersome.

Starting with Lawvere's work, more differential geometry was developed synthetically [Koc06] along with a study of the models of the theory [MR90]. One basic axiom of the theory, the Kock-Lawvere axiom admits intuitive reasoning with nilpotent infinitesimals. We will describe this axiom in the following, since the duality axiom in synthetic algebraic geometry is a generalization. The Kock-Lawvere axiom is added to a basic language which works in good enough categories, for example toposes and admits a couple of basic sets like , $\{*\}$ and $\mathbb{N}$ exist and for objects A, B natural constructions like $A \times B$ or A^B exists and behave as they would for sets – we also have predicates $P(x)$ for elements $x : A$ and can form subobjects like $\{x : A \mid P(x)\}$. In this language, we assume there is a fixed ring R, which can be thought of as the real numbers and if we take the $\mathbb{D}(1) := \{x \in R \mid x^2 = 0\}$ to be the set of all square-zero elements of R, then the Kock-Lawvere axiom gives us a bijection

$$e : R \times R \to R^{\mathbb{D}(1)}$$

which commutes with evaluation at 0 and projection to the first factor. The intuition is that $\mathbb{D}(1)$ is so small that any function on it is linear and therefore determined by its value and its derivative at $0 \in \mathbb{D}(1)$. So with this axiom, the derivative at $0 : R$ of a function $f : R \to R$ may then be defined as $\pi_2(e^{-1}(f_{|\mathbb{D}(1)}))$, which is the start of a convenient way to develop calculus, wihtout defining any further structures on R and other objects – reasoning sythetically means here, that we can just work with these objects as sets.

To give just an example: The tangent bundle of a manifold M can be defined as $M^{\mathbb{D}(1)}$ and vector fields as sections of the canonical map $M^{\mathbb{D}(1)} \to M$. Then it is easy to see that a vector field is the same as a map $\zeta : \mathbb{D}(1) \to M^M$ with $\zeta(0) = \mathrm{id}_M$, which can be interpreted as an infinitesimal transformation of the identity map. This style of reasoning with spaces as if they were sets is also central in current synthetic algebraic geometry and can be quite convenient.

The Kock-Lawvere axiom above and many of the axioms used in synthetic reasoning are incompatible with the law of excluded middle and therefore also with the axiom of choice. It is however a recurring phenomen that restricted versions of excluded middle and choice are compatible with synthetic languages in the sense that they are supported by a model. A very basic example is, that equality of natural numbers is decidable, meaning that two natural numbers are either equal or not equal.

The use of nilpotent elements to capture infinitesimal quantities as mentioned above was inspired by the Grothendieck school of algebraic geometry and Anders Kock also worked with an extended axiom [Koc] suitable for synthetic algebraic geometry, where the role of $\mathbb{D}(1)$ above can be taken by any finitely presented affine scheme. In 2017 Ingo Blechschmidt finished his doctoral thesis in which he noticed a property holding internally in the Zariski-topos, which he called synthetic quasi-coherence – this was a more general and internal verision of what Kock used. In 2018, David Jaz Myers[1] started to work with a specialization of Blechschmidt's synthetic quasi-coherence and used homotopy type theory as a base language, which is the standard in synthetic algebraic geometry now and we will highlight some implications below. Myer's specialized axiom is what we now call *duality axiom*.

To state the duality axiom we need the general concept behind the space $\mathbb{D}(1)$, which is space that are the common zeros of some system of polynomial equations over R. Such a system can be encoded representation independent by a finitely presented R-algebra, i.e. an R-algebra A which is of the form $R[X_1, \ldots, X_n]/(P_1, \ldots, P_l)$ for some numbers n, l and polynomials $P_i \in R[X_1, \ldots, X_n]$. Then the zero set of the system is given by the type $\mathrm{Hom}_{R\text{-Alg}}(A, R)$ of R-algebra homomorphisms from A to the base ring, which we denote by $\mathrm{Spec}\, A$. Now the duality axiom states that Spec is the inverse to exponentiating with R, i.e. for all finitely presented R-algebras A the following is an isomorphism:

$$(a \mapsto (\varphi \mapsto \varphi(a))) : A \to R^{\mathrm{Spec}\, A}.$$

Using homotopy type theory as a language for synthetic algebraic geometry is, in addition to convenience, also a language for synthetic homotopy theory. So instead of the usual practice in algebraic topology to provide model spaces using point-set topology, one can start directly at the level of homotopy types and instead of implementing their higher structure with Kan complexes, there are rules which do not mention any implementation. The rules of homotopy type theory allow to work with the basic objects of the theory, types, in very much the same way as one would work with sets in traditional mathematics – with the clear exception of the law of excluded middle and the axiom of choice - although the former

[1] Myer's never published on the subject, but communicated his ideas to Felix Cherubini and in talks to a larger audience [Mye19b; Mye19a].

and restricted versions of the latter can be assumed. Both can be seen as stating something about the spatial structure. The law of excluded middle allows us to find a complement of each subset of a given set A, which exposes A as a coproduct. This is not true in topology, for example, $\mathbb{R}$ is not the coproduct of the topological subspaces $\{0\}$ and $\mathbb{R}/\{0\}$. The axiom of choice states that any surjection has a section. This is also not true in topology and would trivialize all cohomology. Thus, constructive reasoning in the sense of not using these two axioms is a necessity if we want to use spatial collections in the same way we use sets. In synthetic algebraic geometry, we work inside homotopy type theory and remind readers of this by using the notation $x : X$ which can often be thought of as $x \in X$. [Shu21] is an introduction to homotopy type theory for a general mathematical audience.

One of the main advantages of using specifically homotopy type theory and not a different internal language, is that it is possible to make cohomological computations, using homotopy type theory for synthetic homotopical reasoning. This means that we are mixing two synthetic approaches, combining their advantages, which rests on the possibility of interpreting homotopy type theory in higher toposes [Shu19] and not just the higher topos of ∞-groupoids. The general idea of using homotopy type theory to combine some kind of synthetic, spatial reasoning with synthetic homotopy theory, goes back at least to 2014, to Mike Shulman and Urs Schreiber [SS14]. Schreiber suggested to the HoTT community at various occasions to make use of HoTT as the internal language of higher toposes, where specifities of the topos are accessed in the language via modalities. This approach was shown to be quite effective and intuitive in Shulman's [Shu18] work on mixing synthetic homotopy theory in the form of HoTT and a synthetic approach to topology using a triple of modalities – a structure called cohesion by Lawvere [Law07].

One of Schreiber's motivation was to make use of the modern perspective on cohomology, which in a higher topos can be realized as the connected components of a space of maps. This can be mimicked in HoTT, like follows: Let X be a type and A an abelian group and $n : \mathbb{N}$, then

$$H^n(X, A) := \|X \to K(A, n)\|_0$$

is the n-th cohomology group of X with coefficients in A, where $\|\_\|_0$ is the 0-truncation, an operation which turns a type with possibly non-trivial higher identity types into a set – a type with trivial higher structure. The type $K(A, n)$ is the n-th Eilenberg MacLane space, which can always be constructed for any abelian group A and comes with an isomorphism $\Omega^n(K(A, n)) \simeq A$. This definition of cohomology groups allows using the synthetic homotopy theory to reason about cohomology, which had been already done successfully at the time for the cohomology of homotopy types, like spheres and finite CW-complexes. But, in this case, this kind of reasoning is applied to study 0-types.

In 2022, trying to use this approach to calculate cohomology groups in synthetic algebraic geometry led to the discovery of what is now called Zariski-local choice [CCH24], which is an additional axiom that holds in some cubical models of HoTT based on the Zariski-topos. It is a weakening of the axiom of choice which can be formulated as: For any surjective map $f : X \to Y$, there exists a section, i.e. a map $s : Y \to X$ such that $f \circ s = \mathrm{id}_Y$. Zariski-local choice also states the existence of a section, but only Zariski-local and only for surjections into an affine scheme: For any surjection $f : E \to \operatorname{Spec} A$, there exists a Zariski-cover $U_1, \ldots, U_n$ of $\operatorname{Spec} A$ and maps $s_i : U_i \to E$ such that $f(s_i(x)) = x$ for all $x \in U_i$.

In homotopy type theory, we use the propositional truncation $\|\_\|$ to define surjections and more generally what we mean with "exists". Propositional truncation turns an arbitrary type A into a type $\|A\|$ with the property $x = y$ for all $x, y : \|A\|$. Types with this property are called propositions or (-1)-types in homotopy type theory. Using a univalent universe of types $\mathcal{U}$ we have that surjection into a type A are the same as type families $F : A \to \mathcal{U}$, such that we have $\|F(x)\|$ for all $x : A$. Using type families instead of maps allows us to drop the condition that the maps we get are sections, since we can express it using dependent function types and we arrive at the formulation of Zariski-local choice given below in the list of axioms. In this instance and many others, homotopy type theory provides a lot of convience when working very formally, which is an advantage in formalization of synthetic algebraic geometry.

In total, apart from homotopy type theory and a fixed commutative ring R we use in synthetic algebraic geometry the following three axioms – we will provide some explanation for the first one below:

Axiom (Locality)
R is a local ring, i.e. whenever $x + y$ is invertible x is invertible or y is invertible.

Axiom (Duality)
For any finitely presented R-algebra A, the homomorphism

$$a \mapsto (\varphi \mapsto \varphi(a)) : A \to (\operatorname{Spec} A \to R)$$

is an isomorphism of R-algebras.

Axiom (Zariski-local choice)
Let A be a finitely presented R-algebra and let $B : \operatorname{Spec} A \to \mathcal{U}$ be a family of inhabited types. Then there exists a Zariski-cover $U_1, \ldots, U_n \subseteq \operatorname{Spec} A$ together with dependent functions $s_i : (x : U_i) \to B(x)$.

With the Kock-Lawvere axiom, we introduced the first historic predeccessor of the duality axiom as a strating point for convenient infinitesimal computations, while this is also possible in synthetic algebraic geometry, the general duality axiom has a lot of surprising consequences. In line with classical algebraic geometry, it shows that we have the usual anti-equivalence between finitely presented R-algebras and affine schemes of finite presentation over R. What is more surprising, is the consequence that all functions $R \to R$ are polynomials and that it has implications on the properties of the base ring R. For example, for all $x : R$, x is invertible if and only if we have $x \neq 0$. Duality also implies that affine schemes can only have bounded maps to the natural numbers.

Surprisingly, the Zariski-local choice axiom was also usable to solve problems which have no obvious connection to cohomology. For example, it admits a proof that pointwise open subsets of an affine scheme are the same as subsets which are given by unions of non-vanishing sets of functions on the scheme. In more detail, we say a proposition P is open, if there are a natural number n and elements $r_1, \ldots, r_n$ of the base ring R, such that P is equivalent to the proposition $r_1 \neq 0 \vee \cdots \vee r_n \neq 0$. Then we call a subset U of a type X open, if the proposition $x \in U$ is open for all $x : X$. Using Zariski-local choice, these pointwise existing ring elements can be turned into locally existing functions. For an affine scheme X it is even the case, that an open subset in the pointwise sense, is a union of non-vanishing sets $D(f_i)$ of global functions $f_i : \operatorname{Spec} A \to R$. An analogous result holds also for closed propositions, which are propositions of the form $r_1 = 0 \wedge \cdots \wedge r_n = 0$ for $r_i : R$ and vanishing sets of functions on affine schemes.

The connection between pointwise and local openness is important to make the synthetic definition of a scheme work well: A scheme is a type X, that merely has a finite open cover by affine schemes. To produce interesting examples, it is neccessary to use the locality axiom. This is related to the Zariski topology and ensures that classical examples of Zariski covers can be reproduced. The main example is projective space, which can be defined as the quotient of $R^{n+1}/\{0\}$ by the action of $R^\times$ by scaling. A cover of this type is given by sets of equivalence classes of the form $\{[x_0 : \cdots : x_n] | x_i \neq 0\}$, which is clearly open by the pointwise definition. To see that it is a cover, one has to note that for $x : R^{n+1}$, $x \neq 0$ is equivalent to one of the entries x_i being different from 0. In constructive algebra, this is the case if R is a local ring.

Contribution and organization of the article. We define étale, smooth and unramified schemes and maps in synthetic algebraic geometry in a novel way using what we call *closed dense propositions* (Definition 1.1.2). This is an instance of a general phenomenon in synthetic reasoning, that concepts which are usually defined locally can be defined pointwise or on the level of propositions. While describing the infinitesimal structure of schemes in section 2, we also point out a curious discovery of which we have not found any counterpart in the classical literatures: there is a duality between finitely presented modules over the internal base ring R and finitely copresented modules over the same ring (Lemma 2.3.10 and corollary 2.3.11). The latter notion of finitely copresented modules is not very prominent in algebra, but appears naturally in the study of tangent spaces of schemes (Lemma 2.2.8).

We show that the new definitions agree with straightforward translations of the classical concepts (Remark 1.4.2) and provide some characterizations using tangent spaces: a map between schemes is unramified if and only if it induces injections on all tangent spaces and étale if and only if it induces isomorphisms (Proposition 3.2.1 and corollary 4.1.2). Furthermore, a map from a smooth scheme is smooth if and only if it induces surjections on tangent spaces (Corollary 4.1.1).

Finally, we show that unramified, étale and smooth schemes can be described very concretely in the expected way by conditions on the polynomials locally describing such schemes (Proposition 3.3.2 and lemmas 4.3.3 and 4.3.4). An important intermediate result for the characterization of smooth schemes is that their tangent spaces are finite free R-modules (Proposition 4.2.4)

1 Formally étale, unramified and smooth types

1.1 Definitions

In [CCH24] it is shown that elements of the base ring R are nilpotent if and only if they are not not zero. Both, nilpotency and double negated equality have been used to describe infinitesimals and the following closed dense propositions can be viewed as closed subspaces of the point which are infinitesimally close to being the whole point:

Definition 1.1.1 A closed proposition is dense if it is merely of the form:

$$r_1 = 0 \wedge \cdots \wedge r_n = 0$$

for $r_1, \cdots, r_n : R$ nilpotent.

From a classical perspective, the inclusion $P \subseteq 1$ of a closed dense prosition into the point would be an infinitesimal extension. We will use closed dense propositions in the following to define synthetic analogs of some classical notions, which are classically defined with lifting properties against more general classes of infinitesimal extensions. More details on the connection to classical definitions will be given in Remark 1.4.2.

Definition 1.1.2 A type X is formally étale (resp. formally unramified, formally smooth) if for all closed dense proposition P the map:

$$X \to X^P$$

is an equivalence (resp. an embedding, surjective).

Remark 1.1.3 The map $X \to X^P$ is an equivalence (resp. an embedding, surjective) if and only if for any map $P \to X$ we have a unique (resp. at most one, merely one) dotted lift in:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \\ 1 \arrow[ur, dotted] & \end{tikzcd}$$

Definition 1.1.4 A map is said to be formally étale (resp. formally unramified, formally smooth) if its fibers are formally étale (resp. formally unramified, formally smooth).

Remark 1.1.5 A type (or map) is formally étale if and only if it is formally unramified and formally smooth.

Lemma 1.1.6 A type X is formally étale (resp. formally unramified, formally smooth) if and only if for all $\epsilon : R$ such that $\epsilon^2 = 0$, the map:

$$X \to X^{\epsilon=0}$$

is an equivalence (resp. an embedding, surjective).

Proof The direct direction is obvious as $\epsilon = 0$ is closed dense when $\epsilon^2 = 0$.

For the converse, assume $P = \mathrm{Spec}(R/N)$ a closed dense proposition. Then the map $R \to R/N$ with N finitely generated nilpotent ideal can be decomposed as:

$$R \to A_1 \to \cdots A_n = R/N$$

where A_k is a quotient of R by a finitely generated nilpotent ideal and:

$$A_k \to A_{k+1}$$

is of the form:

$$A \to A/(a)$$

for some $a : A$ with $a^2 = 0$.

We write $P_k = \mathrm{Spec}(A_k)$ and:

$$i_k : P_{k+1} \to P_k$$

so that $\mathrm{fib}_{i_k}(x)$ is $a(x) = 0$ where $a(x)^2 = 0$ holds.

Then by hypothesis we have that for all k and $x : P_k$ the map:

$$X \to X^{\mathrm{fib}_{i_k}(x)}$$

is an equivalence (resp. an embedding, surjective). So the map:

$$X^{P_k} \to \prod_{x:P_k} X^{\mathrm{fib}_{i_k}(x)} = X^{P_{k+1}}$$

is an equivalence (resp. an embedding, surjective by P_k having choice). We conclude that the map:

$$X \to X^P$$

is an equivalence (resp. an embedding, surjective). □

1.2 Stability results

Being formally étale is a modality given as nullification at all dense closed propostions and therefore lex [RSS20, Corollary 3.12]. This means we have the following results:

Proposition 1.2.1

- If X is any type and for all $x : X$ we have a formally étale type Y_x, then:

$$\prod_{x:X} Y_x$$

is formally étale.

- If X is formally étale and for all $x : X$ we have a formally étale type Y_x, then:

$$\sum_{x:X} Y_x$$

is formally étale.

- If X is formally étale then for all $x, y : X$ the type $x = y$ is formally étale.
- The type of formally étale types is formally étale.

Formally unramified type are the separated types [Chr+20, Definition 2.13] associated to formally étale types. By [Chr+20, Lemma 2.15], being formally unramified is a nullification modality as well.

Lemma 1.2.2 A type X is formally unramified if and only if for any $x, y : X$ the type $x = y$ is formally étale.

This means we have the following:

Proposition 1.2.3

- If X is any type and for all $x : X$ we have a formally unramified type Y_x, then:

$$\prod_{x:X} Y_x$$

is formally unramified.

- If X is formally unramified and for all $x : X$ we have a formally unramified type Y_x, then:

$$\sum_{x:X} Y_x$$

is formally unramified.

Being formally smooth is not a modality, indeed we will see it is not stable under identity types. Neverthless we have the following results:

Lemma 1.2.4

- If X is any type satifying choice and for all $x : X$ we have a formally smooth type Y_x, then:

$$\prod_{x:X} Y_x$$

is formally smooth.

- If X is a formally smooth type and for all $x : X$ we have a formally smooth type Y_x, then:

$$\sum_{x:X} Y_x$$

is formally smooth.

1.3 Type-theoretic examples

The next proposition implies that open propositions are formally étale.

Lemma 1.3.1 Any $\neg\neg$-stable proposition is formally étale.

Proof Assume U is a $\neg\neg$-stable proposition. For U to be formally étale it is enough to check that $U^P \to U$ for all P closed dense. This holds because for P closed dense we have $\neg\neg P$. $\square$

Lemma 1.3.2 A closed and formally étale proposition is decidable.

Proof Given a formally étale closed proposition P, let us prove it is $\neg\neg$-stable. Indeed if $\neg\neg P$ then P is closed dense so that $P \to P$ implies P since P is formally étale.

Let I be the finitely generated ideal in R such that:

$$P \leftrightarrow I = 0$$

We have that $I^2 = 0$ implies $\neg\neg(I = 0)$ which implies $I = 0$. But then we have that $I = I^2$, so that by Nakayama (see [LQ15, Lemma II.4.6]) there exists $e : R$ such that $eI = 0$ and $1 - e \in I$. If e is invertible then $I = 0$, if $1 - e$ in invertible then $I = R$. $\square$

Proposition 1.3.3 The type Bool is formally étale.

Proof The identity types in Bool are decidable so Bool is formally unramified. Consider $\epsilon : R$ such that $\epsilon^2 = 0$ and a map:

$$\epsilon = 0 \to \mathrm{Bool}$$

we want to merely factor it through 1.

Since $\mathrm{Bool} \subseteq R$, by duality the map gives $f : R/(\epsilon)$ such that $f^2 = f$. Since $R/(\epsilon)$ is local we conclude that $f = 1$ or $f = 0$ and so the map has constant value $0 : \mathrm{Bool}$ or $1 : \mathrm{Bool}$. $\square$

Remark 1.3.4 This means that formally étale (resp. formally unramified, formally smooth) types are stable by finite sums. In particular finite types are formally étale.

Proposition 1.3.5 The type $\mathbb{N}$ is formally étale.

Proof Identity types in $\mathbb{N}$ are decidable so $\mathbb{N}$ is formally unramified, we want to show it is formally smooth. Assume given a map:

$$P \to \mathbb{N}$$

for P a closed dense proposition, we want to show it merely factors through 1. By boundedness the map merely factors through a finite type, which is formally étale by Remark 1.3.4 so we conclude. $\square$

Lemma 1.3.6 Any proposition is formally unramified.

This means that any subtype of a formally unramified type is formally unramified.

Remark 1.3.7 Given any lex modality, a type is separated if and only if it is a subtype of a modal type, so a type is formally unramified if and only if it is a subtype of a formally étale type.

We also have the following surprising dual result, meaning that any quotient of a formally smooth type is formally smooth:

Proposition 1.3.8 If X is formally smooth and $p : X \to Y$ surjective, then Y is formally smooth.

Proof For any P closed dense and any diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, dotted] & Y \\ 1 \arrow[r, dotted, "x"'] & X \arrow[u, "p"'] \end{tikzcd}$$

by choice for closed propositions we merely get the dotted diagonal, and since X is formally smooth we get the dotted x, and then $p(x)$ gives a lift. $\square$

1.4 Classical definitions, examples and counter-examples

We will show in this section that our definition of étale/smooth/unramified maps and types is equivalent to a internal version of the classical definition. It is important to keep in mind, that our schemes are always locally of finite presentation, so the following definition is sensible:

Definition 1.4.1 A étale (resp. unramified, smooth) type is a scheme which is étale (resp. unramified, smooth). A étale (resp. unramified, smooth) map is a map between schemes which is étale (resp. unramified, smooth).

The following criterion appears as the definition of a formally étale/smooth/unramified map in [EGAIV4, §17], with the only difference, that the lifting property is stated in terms of the comparison map into a pullback and also non-finitely presented algebras and general ideals are considered. The latter is superfluous for maps between schemes, where, as stated above, we only consider schemes that are locally of finite presentation. It is however not clear if this internal criterion corresponds to the (external) definitions in [EGAIV4, §17].

Remark 1.4.2 Let $f : X \to Y$ be a map such that every fiber is a scheme. Then f is formally étale/smooth/unramified if and only if there is exactly one/at least one/at most one lift in all squares

$$\begin{tikzcd} \mathrm{Spec}(A/N) \arrow[r, "t"] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "b"] & Y \end{tikzcd}$$

where A is a finitely presented R-algebra, N a finitely generated nilpotent ideal and the map $\mathrm{Spec}(A/N) \to \mathrm{Spec}(A)$ is induced by the quotient map.

For the proof we will use a cohomological result and the notion of wqc R-modules from [CCH24], to prove the result as stated above – this is would not be neccessary, if we would weaken the statement in the smooth case to Zariski-local existence of lifts.

Proof The inclusion of a closed dense proposition P into 1 is a special case of the left in the remark, so we only need to show, that formally étale, formally smooth and formally unramified maps satisfy the more general lifting property. For formally étale and unramified maps, we can just apply the lifting property for closed dense propositions for all points in $\mathrm{Spec}\,A$. So let $f : X \to Y$ be formally smooth.

Without loss of generality, we can assume N is of the form (a) with a square-zero. Then the existence of a lift can be shown by finding a family of lifts for each point $v : \mathrm{Spec}\,A$ with $\epsilon := a(v)$ and ϕ induced by t:

$$\begin{tikzcd} \epsilon = 0 \arrow[r, "\phi"] \arrow[d] & \mathrm{fib}_f(b(v)) =: X_v \\ 1 \arrow[ur, dotted] & \end{tikzcd}$$

For any two lifts ψ, ξ, we have $\neg\neg(\psi = \xi)$ and can therefore assume $X_v = \mathrm{Spec}\, R[X_1, \dots, X_n]/P_1, \dots, P_l$. We merely have $y : R^n$ such that the dual to ϕ is given by evaluation at y. Then lifts are given by vectors $x : R^n$ such that $P(x) = 0$ and $\epsilon = 0$ implies $x = y$. The latter dualizes to an inclusion of ideals $(x_1 - y_1, \dots, x_n - y_n) \subseteq (\epsilon)$. So we have $x_i - y_i = a_i \epsilon$. Putting everything together, we get the following type of lifts:

$$L_{v,y} := \sum_{\alpha:(\epsilon)^n} dP_y(\alpha) = -P(y)$$

Let $M_{v,y} := \{\alpha : (\epsilon)^n | dP_y(\alpha) = 0\}$, then $M_{v,y}$ is a wqc R-module and L_y has the structure of an $M_{v,y}$-torsor. For a different choice $y' : R^n$, also inducing ϕ, we get $y' = y + \beta$ for some $\beta : (\epsilon)^n$. By computing $P(y' + \alpha)$ in different ways we have:

$$P(y) + dP_y(\beta + \alpha) = P(y + \beta) + dP_{y+\beta}(\alpha) = P(y) + dP_y(\beta) + dP_{y+\beta}(\alpha)$$

And therefore $dP_y(\beta + \alpha) = dP_y(\beta) + dP_{y+\beta}(\alpha)$ and $dP_{y+\beta}(\alpha) = dP_y(\alpha)$, which means that $M_{v,y}$ is independent of y, so let us write M_v instead. We also have:

$$L_{v,y'} = \sum_{\alpha:(\epsilon)^n} dP_y(\beta + \alpha) = -P(y)$$

so translation by β give a isomorphism of M_v-torsors. So we get a well-defined M_v-torsor L_v by Krauss' Lemma for 1-types. Then by [CCH24], $H^1(v : \operatorname{Spec} A, M_v)$ vanishes and we therefore have a global lift.□

The following lemma is easy to proof – we conclude this section afterwards with counter examples for smoothness and one example of an étale scheme.

Lemma 1.4.3 For all $k : \mathbb{N}$, we have that $\mathbb{A}^k$ is smooth.

Proof Let P be a closed dense proposition and N a nilpotent, finitely generated ideal such that $P = \operatorname{Spec} R/N$. Then $\operatorname{Spec} R[X_1, \ldots, X_k] = \mathbb{A}^k$ is smooth lifts always exist as indicated below by the universal property of $R[X_1, \ldots, X_k]$:

$$\begin{tikzcd} R/N & R[X_1, \ldots, X_k] \arrow[l] \arrow[dl, dotted] \\ R \arrow[u] & \end{tikzcd}$$

□

Example 1.4.4 The affine scheme $\operatorname{Spec}(R[X]/X^2)$ is not smooth.

Proof If it were smooth then, for any ϵ with $\epsilon^3 = 0$, we would be able to prove $\epsilon^2 = 0$. Indeed we would merely have a dotted lift in:

$$\begin{tikzcd} R/(\epsilon^2) & R[X]/(X^2) \arrow[l, "\epsilon"'] \arrow[dl, dashed] \\ R \arrow[u] & \end{tikzcd}$$

that is, an $r : R$ such that $(\epsilon + r\epsilon^2)^2 = 0$. Then $\epsilon^2 = 0$. □

Example 1.4.5 The affine scheme $\operatorname{Spec}(R[X, Y]/XY)$ is not smooth.

Proof Again, we assume a lift for any ϵ with $\epsilon^3 = 0$:

$$\begin{tikzcd} R/(\epsilon^2) & R[X, Y]/(XY) \arrow[l] \arrow[dl, dashed] \\ R \arrow[u] & \end{tikzcd}$$

where the top map sends both X and Y to ϵ. Then we have $r, r' : R$ such that $(\epsilon + r\epsilon^2)(\epsilon + r'\epsilon^2) = 0$ so that $\epsilon^2 = 0$. □

We will proof a generalization of the following example in Lemma 4.3.3. The essential step is to improve a zero $g(y) = 0$ up to some square-zero ϵ to an actual zero.

Example 1.4.6 Let g be a polynomial in $R[X]$ such that for all $x : R$ we have that $g(x) = 0$ implies $g'(x) \neq 0$. Then:

$$\operatorname{Spec}(R[X]/g)$$

is étale.

1.5 Being formally étale, unramified or smooth is Zariski local

Lemma 1.5.1 Let X with $(U_i)_{i:I}$ be a finite open cover of X. Then X is formally étale (resp. formally unramified, formally smooth) if and only if all the U_i are formally étale (resp. formally unramified, formally smooth).

Proof First, we show this for formally unramified:

- Any subtype of a formally unramified type is formally unramified by Lemma 1.3.6.
- Conversely, assume X with such a cover, for all $x, y : X$ there exists $i : I$ such that $x \in U_i$ and then:

$$x =_X y \leftrightarrow \sum_{y \in U_i} x =_{U_i} y$$

which is formally étale because open propositions are formally étale by Lemma 1.3.1.

Now for formally smooth:

- Open propositions are formally smooth by Lemma 1.3.1 so that open subtypes of formally smooth types are formally smooth.
- Conversely if each U_i is formally smooth then $\Sigma_{i:I}U_i$ is formally smooth by Remark 1.3.4, so we can conclude by applying Proposition 1.3.8 to the surjection:

$$\Sigma_{i:I}U_i \to X$$

The result for formally étale immediately follows. □

Corollary 1.5.2 For all $k : \mathbb{N}$, the projective space $\mathbb{P}^k$ is smooth.

Proof By Lemma 1.5.1 it is enough to check that $\mathbb{A}^k$ is smooth. This is Lemma 1.4.3 □

2 Linear algebra and tangent spaces

2.1 Modules and infinitesimal disks

The most basic infinitesimal schemes are the first order neighbourhoods in affine n-space R^n. Their algebra of functions is R^{n+1}, which is an instance of the more general construction below.

For any R-module M, there is an R-algebra structure on $R \oplus M$ with multiplication given by

$$(r, m)(r', m') = (rr', rm' + r'm)$$

Algebras of this form are called *square zero extensions* of R, since products of the form $(0, m)(0, n)$ are zero. By this property, for any R-linear map $\varphi : M \to N$ between modules M, N, the map $\mathrm{id} \oplus \varphi : R \oplus M \to R \oplus N$ is an R-algebra homomorphism. In particular, if M is finitely presented, i.e. merely the cokernel of some $p : R^n \to R^m$ then $R \oplus M$ is the cokernel of a map between finitely presented algebras and therefore finitely presented as an algebra.

Definition 2.1.1 Given M a finitely presented R-module, we define a finitely presented algebra structure on $R \oplus M$ as above and set:

$$\mathbb{D}(M) :\equiv \mathrm{Spec}(R \oplus M)$$

This is a pointed scheme by the first projection which we denote 0 and the construction is functorial by the discussion above.

We write $\mathbb{D}(n)$ for $\mathbb{D}(R^n)$ so that for example:

$$\mathbb{D}(1) = \mathrm{Spec}(R[X]/(X^2)) = \{\epsilon : R \mid \epsilon^2 = 0\}$$

Definition 2.1.2 Assume given M a finitely presented R-module and A a finitely presented R-algebra with $x : \mathrm{Spec}(A)$. An M-derivation at x is a morphism of R-modules:

$$d : A \to M$$

such that for all $a, b : A$ we have that:

$$d(ab) = a(x)d(b) + b(x)d(a)$$

Lemma 2.1.3 Assume given M a finitely presented module and A a finitely presented algebra with $x : \mathrm{Spec}(A)$. Pointed maps:

$$\mathbb{D}(M) \to_{\mathrm{pt}} (\mathrm{Spec}(A), x)$$

correspond to M-derivations at x.

Proof Such a pointed map correponds to an algebra map:

$$f : A \to R \oplus M$$

where the composite with the first projection is x. This means that, for some module map $d : A \to M$ we have:

$$f(a) = (a(x), d(a))$$

We can immediately see that f being a map of R-algebras is equivalent to d being an M-derivation at x. □

Lemma 2.1.4 Let M, N be finitely presented modules. Then linear maps $M \to N$ correspond to pointed maps $\mathbb{D}(N) \to_{\text{pt}} \mathbb{D}(M)$.

Proof By Lemma 2.1.3 such a pointed map corresponds to an N-derivation at $0 : \mathbb{D}(M)$.

Such a derivation is a morphism of modules:

$$d : R \oplus M \to N$$

such that for all $(r, m), (r', m') : R \oplus M$ we have that:

$$d(rr', rm' + r'm) = rd(r', m') + r'd(r, m)$$

This implies $d(r, 0) = 0$ for all $r : R$, so we obtain a section to the injective functorial action of $\mathbb{D}$ on linear maps. □

2.2 Tangent spaces

Definition 2.2.1 Let X be a type and let $x : X$, then we define the *tangent space* $T_x(X)$ of X at x by:

$$\{t : \mathbb{D}(1) \to X \mid t(0) = x\}$$

Definition 2.2.2 Given $f : X \to Y$ and $x : X$ we have a map:

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

induced by post-composition.

Lemma 2.2.3 For all $x : R^n$ we have $T_x(R^n) = R^n$.

Proof Since R^n is homogeneous we can assume $x = 0$. By Lemma 2.1.3 we know that $T_0(R^n)$ corresponds to the type of linear maps

$$R[X_1, \cdots, X_n] \to R$$

such that for all P, Q we have:

$$d(PQ) = P(0)dQ + Q(0)dP$$

which is equivalent to $d(1) = 0$ and $d(X_iX_j) = 0$, so any such map is determined by its image on the X_i so it is equivalent to an element of R^n. □

Lemma 2.2.4 Given a scheme X with $x : X$ and $v, w : T_x(X)$, there exists a unique:

$$\psi_{v,w} : \mathbb{D}(2) \to_{\text{pt}} X$$

such that for all $\epsilon : \mathbb{D}(1)$ we have that:

$$\psi_{v,w}(\epsilon, 0) = v(\epsilon)$$
$$\psi_{v,w}(0, \epsilon) = w(\epsilon)$$

Proof We can assume X is affine. Then $\mathbb{D}(2) \to_{\text{pt}} X$ is equivalent to the type of R^2-derivations at x, but giving an $M \oplus N$-derivation is equivalent to giving an M-derivation and an N-derivation. Checking the equalities is a routine computation. □

Lemma 2.2.5 For any scheme X and $x : X$, we have that $T_x(X)$ is a module.

Proof There is a more conceptual proof given as [Mye22, Theorem 4.2.19] which could be made to work with schemes – we proceed by sketching a more tedious, explicit proof with less technical prerequisites. We define scalar multiplication by sending v to $t \mapsto v(rt)$.

Then for addition of $v, w : T_x(X)$, we define:

$$(v + w)(\epsilon) = \psi_{v,w}(\epsilon, \epsilon)$$

where $\psi_{v,w}$ is defined in Lemma 2.2.4.

We omit checking that this is a module structure. □

Lemma 2.2.6 For $f : X \to Y$ a map between schemes, for all $x : X$ the map df_x is a map of R-modules.

Proof Commutation with scalar multiplication is immediate.

Commutation with addition comes by applying uniqueness in Lemma 2.2.4 to show:

$$f \circ \psi_{v,w} = \psi_{f \circ v, f \circ w} \qquad \square$$

Lemma 2.2.7 For any map $f : X \to Y$ and $x : X$, we have that:

$$\mathrm{Ker}(df_x) = T_{(x, \mathrm{refl}_{f(x)})}(\mathrm{fib}_f(f(x)))$$

Proof This holds because:

$$(\mathrm{fib}_f(f(x)), (x, \mathrm{refl}_{f(x)}))$$

is the pullback of:

$$(X, x) \to (Y, f(x)) \leftarrow (1, *)$$

in pointed types, applied using $(\mathbb{D}(1), 0)$. $\square$

Lemma 2.2.8 Let X be a scheme with $x : X$. Then $T_x(X)$ is a finitely copresented R-module.

Proof We can assume X affine. For some map $P : R^m \to R^n$ we have $X = \mathrm{fib}_P(0)$. By applying Lemma 2.2.7 we know that $T_x(X)$ is the kernel of $dP_x : T_x(R^m) \to T_0(R^n)$ for all $x : X$. We conclude by Lemma 2.2.3. $\square$

Corollary 2.2.9 Let X be a scheme, then the tangent bundle $X^{\mathbb{D}(1)}$ is a scheme.

Proof We give two independent arguments, the first uses the lemma, the second is a direct computation:

(i) Finitely copresented modules are schemes, since they are the common zeros of linear functions on R^n. So by the lemma, all tangent spaces $T_x(X)$ are schemes and

$$X^{\mathbb{D}(1)} = \sum_{x:X} T_x(X)$$

is a dependent sum of schemes and therefore a scheme.

(ii) Let X be covered by open affine $U_1, \dots, U_n$ then $U_1^{\mathbb{D}(1)}, \dots, U_n^{\mathbb{D}(1)}$ is an open cover by double negation stability of opens. So we conclude by showing that for affine $Y = \mathrm{Spec}\, R[X_1, \dots, X_n]/(f_1, \dots, f_l)$ the tangent bundle $Y^{\mathbb{D}(1)}$ is affine by direct computation:

$$\begin{aligned} Y^{\mathbb{D}(1)} &= \mathrm{Hom}_{R\text{-Alg}}(R[X_1, \dots, X_n]/(f_1, \dots, f_l), R \oplus \epsilon R) \\ &= \{(y_1, \dots, y_n) : R \oplus \epsilon R \mid \forall_i . f_i(y_1, \dots, y_n) = 0\} \\ &= \{(x_1, \dots, x_n, d_1, \dots, d_n) : R^{2n} \mid \forall_i . f_i(x_1, \dots, x_n) = 0 \text{ and } \sum_j d_j \frac{\partial f_i}{\partial X_j}(x_1, \dots, x_n) = 0\} \end{aligned} \qquad \square$$

Definition 2.2.10 For X a type and $p : X$ a point, the *cotangent space* at p is the R-linear dual $T_p X^\star$ of the tangent space $T_p X$.

If X is a scheme, then by lemma 2.2.8 the cotangent spaces of X are finitely presented. We will not use the following definition and remark in the rest of this article, but included them to show a connection to the classical theory. See [Har77, p. 172] or [Vak, p. 573] for the classical theory.

Definition 2.2.11 For A an R-module, there is a universal derivation $d : A \to \Omega_{A/R}$. We call elements of the type $\Omega_{A/R}$ *Kähler differentials*.

That is, $\Omega_{A/R}$ is generated as an A-module by symbols df for $f : A$, subject to relations $d = r \cdot df$ for $r : R$ and $d(fg) = f \cdot dg + g \cdot df$. It can be seen that if A is finitely presented as an R-algebra, then $\Omega_{A/R}$ is finitely presented as an A-module. Classically, the sheaf $\Omega_{A/R}$ is the cotangent bundle. Synthetically, it is enough to show this pointwise on $\mathrm{Spec}\, A$ by [CCH24, Theorem 8.2.3]. To apply this, we first turn $\Omega_{A/R}$ into an R-module bundle on $\mathrm{Spec}\, A$: For $p : \mathrm{Spec}\, A$, let $\Omega_{A/R,p}$ be the type of R-derivations at p, as defined in Definition 2.1.2 – this agrees with tensoring $\Omega_{A/R,p}$ with R using the evaluation at p, which is the general construction used in [CCH24, Theorem 8.2.3].

Remark 2.2.12 For all $p : \operatorname{Spec} A$, we have $\Omega_{A/R,p} \simeq T_pX^\star$ and therefore also $\Omega_{A/R} \simeq \prod_{x:\operatorname{Spec} A} T_pX^\star$.

Proof We need to show that for $p : X$, we have an isomorphism of R-modules

$$T_p^\star X \simeq \Omega_{A/R,p} = \Omega_{A/R} \otimes_A R.$$

By Lemma 2.1.3 the tangent space T_pX corresponds to derivations $A \to R$, where the A-module structure on R is obtained by evaluating at p. These derivations correspond to A-module maps $\Omega_{A/R} \to R$, by the universal property of Kähler differentials. In Corollary 2.3.11, we will see that double dualization is an isomorphism for finitely (co-)presented R-modules, so we can conclude by dualizing. □

2.3 Infinitesimal neighbourhoods

Definition 2.3.1 Let X be a set with $x : X$. The *first order neighborhood* $N_1(x)$ is defined as the set of $y : X$ such that there exists an finitely generated ideal $I \subseteq R$ with $I^2 = 0$ and:

$$I = 0 \to x = y$$

Lemma 2.3.2 Assume $x, y : R^n$, then $x \in N_1(y)$ if and only if the ideal generated by the $x_i - y_i$ squares to zero.

Proof Let us denote I the ideal generated by the $x_i - y_i$ so that $x = y$ if and only if $I = 0$.

If $I^2 = 0$ then it is clear that $y \in N_1(x)$.

Conversely if $J = 0 \to I = 0$ then we have that $I \subset J$ by duality so that $J^2 = 0$ implies $I^2 = 0$. □

Lemma 2.3.3 Let X be a scheme with $x : X$. Then $N_1(x)$ is an affine scheme.

Proof If $x \in U$ open in X, we have that $N_1(x) \subset U$ so that we can assume X affine.

This means X is a closed subscheme $C \subset R^n$. Then by Lemma 2.3.2, we have that $N_1(x)$ is the type of $y : R^n$ such that $y \in C$ and for all i, j we have that $(x_i - y_i)(x_j - y_j) = 0$, which is a closed subset of C so it is an affine scheme. □

Definition 2.3.4 A pointed scheme $(X, *)$ is called a *first order (infinitesimal) disk* if for all $x : X$ we have $x \in N_1(*)$.

Lemma 2.3.5 N_1 extends to a functor from pointed schemes to first order disks.

Proof Since for $x : X$, $N_1(x)$ is just a subspace the functoriality is clear once we know that the defining relation of first order disks is preserved by functions between schemes. It is enough to consider $f : X \to Y$ for affine $X, Y \subseteq R^n$. So let $x, y : X$ such that $y \in N_1(x)$. Then for $\epsilon := x - y$ we have $e_ie_j = 0$ for all i, j and

$$f(x) - f(y) = f(x) - f(x + \epsilon) = f(x) - (f(x) + df_x\epsilon) = -df_x\epsilon$$

which means that $f(y) \in N_1(f(x))$. □

Lemma 2.3.6 A pointed scheme $(X, *)$ is a first order disk if and only if there exists a finitely presented module M such that:

$$(X, *) = (\mathbb{D}(M), 0)$$

Proof First we check that for all M finitely presented and $y : \mathbb{D}(M)$ we have that $y \in N_1(0)$. Let $m_1, \cdots, m_n$ be generators of M, then consider $d : M \to R$ induced by y, then $y = 0$ if and only if $d = 0$ and for all i, j we have that:

$$d(m_i)d(m_j) = 0$$

This means that $I = (d(m_1), \cdots, d(m_n))$ has square 0 and $I = 0$ implies $y = 0$ so that $y \in N_1(0)$.

For the converse we assume X a first order disk, by Lemma 2.3.3 we have that X is affine and pointed, up to translation we can assume $X \subset R^n$ closed pointed by 0. Since X is a first order disk we have that $X \subset N_1(0)$ and by Lemma 2.3.2 we have $N_1(0) = \mathbb{D}(R^n)$.

This means there is an f.g. ideal J in $R \oplus R^n$ such that $X = \operatorname{Spec}(R \oplus R^n/J)$. But 0 corresponds to the first projection from $R \oplus R^n$, so that $0 \in X$ means that if $(x, y) \in J$ then $x = 0$, so that J corresponds uniquely to an f.g. sub-module K of R^n and:

$$X = \operatorname{Spec}(R \oplus (R^n/K)) = \mathbb{D}(R^n/K)$$ □

Definition 2.3.7 Let $M^\star :\equiv \mathrm{Hom}_R(M, R)$ denote the R-linear dual of an R-module M.

While it is clear that the dual of a finite presentation yields a finite copresentation, the reverse is not true in general, but holds with the duality axiom. We will give a proof of this fact in Lemma 2.3.10, which will need the following two extension results.

Lemma 2.3.8 Let $M \subseteq R^n$ be the kernel of a linear map between finite free R-modules. Then any linear map $M \to R$ can be extended to R^n.

Proof First note that $M = \mathrm{Spec}(R[X_1, \dots, X_n]/(l_1, \dots, l_m))$ is affine, where the l_i are linear. Let $L : M \to R$ be R-linear and $P : R^n \to R$ be given by taking a preimage under the quotient map $R[X_1, \dots, X_n] \to R[X_1, \dots, X_n]/(l_1, \dots, l_m)$, so we have $P_{|M} = L$. Let $P = \sum_{\sigma:\mathbb{N}^{\{1,\dots,n\}}} a_\sigma X_1^{\sigma(1)} \cdots X_n^{\sigma(n)}$. Now we can conclude by showing that the linear part of P

$$K :\equiv \sum_{\sigma:\mathbb{N}^{\{1,\dots,n\}}, \sum\sigma=1} a_\sigma X_1^{\sigma(1)} \cdots X_n^{\sigma(n)}$$

extends L as well, i.e. we will see $K_{|M} = L$.

For all $x : M$ and $\lambda : R$ we have $L(\lambda x) = \lambda L(x)$ and therefore

$$\sum_{\sigma:\mathbb{N}^{\{1,\dots,n\}}} \lambda^{\sum\sigma} a_\sigma x_1^{\sigma(1)} \cdots x_n^{\sigma(n)} = \lambda \sum_{\sigma:\mathbb{N}^{\{1,\dots,n\}}} a_\sigma x_1^{\sigma(1)} \cdots x_n^{\sigma(n)}$$

By comparing coefficients as polynomials in λ, we have $\sum_{\sigma:\mathbb{N}^{\{1,\dots,n\}}, \sum\sigma\neq 1} a_\sigma x_1^{\sigma(1)} \cdots x_n^{\sigma(n)} = 0$, which shows $K_{|M} = P_{|M} = L$. $\square$

Lemma 2.3.9 Let $\varphi : R^n \to R^m$ be R-linear, then any linear map $\mathrm{im}(\varphi) \to R$ on the image of φ can be extended to R^m.

Proof [2] Let $(a_{ij})_{ij}$ be the coefficients of the matrix representing φ with respect to the standard basis. Then the image of φ is generated by the columns of this matrix:

$$\mathrm{im}(\varphi) = \left\{ \sum_{j=1}^{n} x_j (a_{\_j}) \mid x_j : R,\, 1 \leq j \leq n \right\}$$

Let $L : \mathrm{im}(\varphi) \to R$ be R-linear and $l_j :\equiv L((a_{\_j}))$. Applying L to a general element of $\mathrm{im}(\varphi)$ and using linearity yields the following implication:

$$\sum_{j=1}^{n} x_j (a_{\_j}) = 0 \Rightarrow \sum_{j=1}^{n} x_j l_j = 0$$

The left side being 0 means that m linear polynomials $P_i(x_1, \dots, x_n) :\equiv \sum_{j=1}^{n} x_j a_{ij}$ vanish simulataneously. Let $Q(x_1, \dots, x_n)$ be the linear polynomial on the right side of the implication. Then the implication induces an inclusion between the common zeros of the P_i and the zeros of Q, which by duality means that we have an inclusion of ideals $(Q) \subseteq (P_1, \dots, P_m)$ in $R[X_1, \dots, X_n]$. So there $b_i : R[X_1, \dots, X_n]$ such that

$$Q = \sum_{i=1}^{m} b_i P_i$$

By comparing coefficients it is clear that the b_i can be chosen to be in R, which we assume now. We

[2] This proof is due to Thierry Coquand.

define a R-linear map $K : R^m \to R$ by $K((y_1, \dots, y_m)^T) :\equiv \sum_{i=1}^m b_i y_i$. K extends L:

$$\begin{aligned}
K\left(\sum_{j=1}^n x_j(a_{_j})\right) &= \sum_{i=1}^m b_i \sum_{j=1}^n x_j a_{ij} \\
&= \sum_{i=1}^m b_i P_i(x_1, \dots, x_n) \\
&= Q(x_1, \dots, x_n) \\
&= \sum_{j=1}^n x_j l_j \\
&= L\left(\sum_{j=1}^n x_j(a_{_j})\right)
\end{aligned}$$

□

Lemma 2.3.10 Let M be finitely copresented, i.e. let there be an exact sequence

$$M \xhookrightarrow{\varphi} R^n \xrightarrow{P} R^m$$

Then the dual of this sequence is exact as well and $\varphi^\star$ is surjective. In particular, $M^\star$ is finitely presented.

Proof Surjectivity of $\varphi^\star$ follows from Lemma 2.3.8. Linear maps $R^n \to R$ which vanish on M factor over the image of P, so exactness at the middle of the dual sequence follows from Lemma 2.3.9. □

Corollary 2.3.11 For any module M finitely presented or finitely copresented, we have that $M^{\star\star} = M$.

Lemma 2.3.12 The functor from finitely copresented modules to first order disks:

$$M \mapsto \mathbb{D}(M^\star)$$

is an equivalence, with inverse:

$$(X, x) \mapsto T_x(X)$$

Proof It is fully faithful by Lemma 2.1.4 and essentially surjective by Lemma 2.3.6. To check for the inverse it is enough to check that:

$$T_0(\mathbb{D}(M^\star)) = M$$

But by Lemma 2.1.4 we have that $T_0(\mathbb{D}(M^\star)) = M^{\star\star}$ and we conclude by Corollary 2.3.11. □

Lemma 2.3.13 Let X be a scheme with $x : X$, then we have:

$$N_1(x) = \mathbb{D}(T_x(X)^\star)$$

Proof By Lemma 2.3.3 we have that $(N_1(x), x)$ is a first order disk. By Lemma 2.3.12 it is enough to check that $T_x(N_1(x)) = T_0(\mathbb{D}(T_x(X)^\star))$.

It is immediate that any map $f : \mathbb{D}(1) \to X$ uniquely factors through $N_1(f(0))$ so that $T_x(N_1(x)) = T_x(X)$, and we have that $T_0(\mathbb{D}(T_x(X)^\star)) = T_x(X)$ by Lemma 2.3.12. □

2.4 Projectivity of finitely copresented modules

Finitely copresented R-modules are projective objects in the category of finitely copresented R-modules, which means that all surjections between finitely copresented R-modules split. The results in this section will not be used in the rest of the article.

Lemma 2.4.1 Let M be a finitely copresented module, then we have $T_0(M) = M$.

Proof We have that M is the kernel of a linear map $P : R^m \to R^n$. By Lemma 2.2.7 we have that $T_0(M)$ is the kernel of:

$$dP_0 : T_0(R^m) \to T_0(R^n)$$

but by Lemma 2.2.3 this is a map from R^m to R^n, we omit the verification that $dP_0 = P$. □

Lemma 2.4.2 Any finitely copresented module is projective.

Proof We consider M, N finitely copresented with a surjective map:

$$f : M \to N$$

By Lemma 2.3.13 and Lemma 2.4.1 we know that $\mathbb{D}(M^*) = N_1(0)$ in M, so that we have a commutative diagram:

$$\begin{tikzcd} \mathbb{D}(M^\star) \arrow[r] \arrow[d] & M \arrow[d, "f"] \\ \mathbb{D}(N^\star) \arrow[r, "i"] & N \end{tikzcd}$$

Since $\mathbb{D}(N^*)$ has choice and f is surjective there is $g : \mathbb{D}(N^\star) \to M$ such that $f \circ g = i$. Up to translation we can assume $g(0) = 0$. Then we can factor g through $\mathbb{D}(M^\star)$ as N_1 is functorial by Lemma 2.3.5. This gives us a pointed section of the map:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

which by Lemma 2.1.4 gives a linear section of f. □

Lemma 2.4.3 A linear map between finitely copresented module:

$$f : M \to N$$

is surjective if and only if the corresponding pointed map:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

merely has a section preserving 0.

Proof By Lemma 2.1.4 we know that:

$$\mathbb{D}(M^\star) \to \mathbb{D}(N^\star)$$

merely having a section preserving 0 is equivalent to:

$$f : M \to N$$

merely having a section. But since any finitely copresented module is projective, this is equivalent to f being surjective. □

3 Formally unramified schemes

3.1 Unramified schemes

Lemma 3.1.1 Let X be an affine scheme, the following are equivalent:

(i) X is formally unramified.

(ii) Identity types in X are decidable.

(iii) For all $x : X$, we have that $T_x(X) = 0$.

Proof (i) implies (ii): By Lemma 1.3.2.

(ii) implies (i): Decidable propositions are formally étale.

(ii) implies (iii): Assume given $x : X$ with $t : T_x(X)$, then for all $\epsilon : \mathbb{D}(1)$ we have $\neg\neg(\epsilon = 0)$ so that we have $\neg\neg(t(\epsilon) = t(0))$ which implies $t(\epsilon) = t(0)$ since equality is assumed decidable. Therefore $t = 0$ in $T_x(X)$.

(iii) implies (i): Indeed given $\epsilon : R$ such that $\epsilon^2 = 0$, assume $x, y : X$ such that $\epsilon = 0 \to x = y$. Then $x \in N_1(y)$ and by Lemma 2.3.13 and $T_y(X) = 0$ we conclude $x = y$. □

Corollary 3.1.2 Let X be a scheme, the following are equivalent:

(i) X is formally unramified.

(ii) Identity types in X are open.

(iii) For all $x : X$, we have that $T_x(X) = 0$.

Proof Assume $(U_i)_{i:I}$ a finite cover of X by affine schemes. By Lemma 1.5.1 we have that X is formally unramified if and only if U_i is formally unramified for all $i : I$.

(ii) implies (i). By Lemma 1.3.1.

(i) implies (iii). Indeed for all $x : X$ there exists $i : I$ such that $x \in U_i$ and then $T_x(X) = T_x(U_i)$ and $T_x(U_i) = 0$ by Lemma 3.1.1.

(iii) implies (ii). Assume $x, y : X$, then:

$$x =_X y \leftrightarrow \Sigma_{y \in U_i} x =_{U_i} y$$

By Lemma 3.1.1 we have that identity types in U_i is decidable, so $x =_X y$ is open. □

3.2 Unramified morphisms between schemes

Now we generalise this to maps between schemes.

Proposition 3.2.1 A map between schemes is unramified if and only if its differentials are injective.

Proof The map df_x is injective if and only if its kernel is 0. By Lemma 2.2.7, this means that df_x is injective for all $x : X$ if and only if:

$$\prod_{x:X} T_{(x,\mathrm{refl}_{f(x)})}(\mathrm{fib}_f(f(x))) = 0$$

On the other hand having fibers with trivial tangent space is equivalent to:

$$\prod_{y:Y} \prod_{x:X} \prod_{p:f(x)=y} T_{(x,p)}(\mathrm{fib}_f(y)) = 0$$

Both are equivalent by path elimination on p. □

3.3 Unramified schemes are locally standard

Definition 3.3.1 A scheme is called standard unramified if it is of the form:

$$\mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_k)$$

with $k \geq n$ such that the determinant of:

$$\left(\frac{\partial P_i}{\partial X_j}\right)_{1 \leq i,j \leq n}$$

is invertible.

Proposition 3.3.2 A scheme is unramified if and only if it has a cover by standard unramified schemes.

Proof By Lemma 1.5.1, it is enough to consider an affine scheme $X = \mathrm{Spec}(R[X_1, \cdots, X_n]/P_1, \cdots, P_k)$. For any $x : X$, by Lemma 2.2.7 we have an exact sequence:

$$0 \longrightarrow T_x(X) \longrightarrow R^n \longrightarrow R^k$$

By Lemma 3.1.1, X is unramified if and only if $T_x(X) = 0$ for all $x : X$.

We have $T_x(X) = \ker(\mathrm{Jac}(P_1, \cdots, P_k)_x)$, so that $T_x(X) = 0$ if and only if $n \leq k$ and this Jacobian has an invertible n-minor. The latter is the case for a standard unramified scheme and the converse follows by covering according to which n-minor is invertible and reordering variables. □

4 Formally smooth and étale schemes

4.1 Smooth and étale maps between schemes

Note that it is immediate from the definition of smoothness that smooth maps induce surjections on tangent spaces. We have a converse when the domain is smooth.

Corollary 4.1.1 Let $f : X \to Y$ be a map between schemes with X smooth. Then the following are equivalent:

(i) The map f is smooth.

(ii) For all $x : X$, the induced map:
$$df : T_x(X) \to T_{f(x)}(Y)$$
is surjective.

Proof (i) implies (ii). Assume given a map $v : \mathbb{D}(1) \to Y$ such that $v(0) = f(x)$, then for all $t : \mathbb{D}(1)$ we have a map:
$$t = 0 \to \mathrm{fib}_f(v(t))$$
so since f is smooth we merely have $w_t : \mathrm{fib}_f(v(t))$ such that $t = 0$ implies $w_t = 0$. We conclude using choice over $\mathbb{D}(1)$.

(ii) implies (i). Assume given $y : Y$ and $\epsilon : R$ such that $\epsilon^2 = 0$ and try to merely find a dotted lift in:

$$\begin{tikzcd}
\epsilon = 0 \arrow[r, "\phi"] \arrow[d] & \mathrm{fib}_f(y) \\
1 \arrow[ur, dashed] &
\end{tikzcd}$$

Since X is formally smooth we merely have an $x : X$ such that:
$$\prod_{p:\epsilon=0} \phi(p) = x$$
and therefore:
$$\epsilon = 0 \to y = f(x)$$
which means that $y \in N_1(f(x))$. We use Lemma 2.4.3 to get that the map $N_1(x) \to N_1(f(x))$ induced by f merely has a section s sending $f(x)$ to x.

Then $s(y) : \mathrm{fib}_f(y)$ such that for all $p : \epsilon = 0$ we have that:
$$\phi(p) = x = s(f(x)) = s(y)$$
□

Corollary 4.1.2 Let $f : X \to Y$ be a map between schemes. Assume X is smooth. Then the following are equivalent:

(i) The map f is étale.

(ii) For all $x : X$, the induced map:
$$df : T_x(X) \to T_{f(x)}(Y)$$
is an iso.

Proof We apply Proposition 3.2.1 and Corollary 4.1.1. □

4.2 Smooth schemes have free tangent spaces

Lemma 4.2.1 Assume X is a smooth scheme. Then for any $x : X$ the type $T_x(X)$ is formally smooth.

Proof Consider $T(X) = X^{\mathbb{D}(1)}$ the total tangent bundle of X. We have to prove that the map:

$$p : T(X) \to X$$

is formally smooth. Both source and target are schemes, and the source is formally smooth because X is smooth and $\mathbb{D}(1)$ has choice. So by Corollary 4.1.1 it is enough to prove that for all $x : X$ and $v : T_x(X)$ the induced map:

$$dp : T_{(x,v)}(T(X)) \to T_x(X)$$

is surjective.

Consider $u : T_x(X)$. By unpacking the definition of tangent spaces and computing $dp(w)$, we see that merely finding $w : T_{(x,v)}(T(X))$ such that $dp(w) = u$ means merely finding:

$$\phi : \mathbb{D}(1) \times \mathbb{D}(1) \to X$$

such that for all $t : \mathbb{D}(1)$ we have that:

$$\phi(0, t) = v(t)$$
$$\phi(t, 0) = u(t)$$

But we know that there exists a unique:

$$\psi_{v,u} : \mathbb{D}(2) \to X$$

such that:

$$\psi_{v,u}(0, t) = v(t)$$
$$\psi_{v,u}(t, 0) = u(t)$$

as defined in Lemma 2.2.4.

Then the fact that X is smooth and that the fibers of:

$$\mathbb{D}(2) \to \mathbb{D}(1) \times \mathbb{D}(1)$$

are closed dense with $\mathbb{D}(1)\times\mathbb{D}(1)$ having choice means that there merely exists a lift of $\psi_{v,w}$ to $\mathbb{D}(1)\times\mathbb{D}(1)$, which gives us the ϕ we wanted. $\square$

Lemma 4.2.2 Assume given a linear map:

$$M : R^m \to R^n$$

which has a formally smooth kernel K. Then we can decide whether $M = 0$.

Proof Since $M = 0$ is closed, it is enough to prove that it is $\neg\neg$-stable to conclude that it is decidable by Lemma 1.3.1 and Lemma 1.3.2. Assume $\neg\neg(M = 0)$, then for any $x : R^m$ we have a dotted lift in:

$$\begin{tikzcd} M = 0 \arrow[r, "\_\mapsto x"] \arrow[d] & K \\ 1 \arrow[ur, dotted] & \end{tikzcd}$$

because K is formally smooth, so that we merely have $y : K$ such that:

$$M = 0 \to x = y$$

which implies that $\neg\neg(x = y)$ since we assumed $\neg\neg(M = 0)$.

Then considering a basis $(x_1, \cdots, x_n)$ of R^m, we get $(y_1, \cdots, y_n)$ such that for all i we have that $M(y_i) = 0$ and $\neg\neg(y_i = x_i)$. But then we have that $(y_1, \cdots, y_n)$ is infinitesimally close to a basis and that being a basis is an open proposition, so that $(y_1, \cdots, y_n)$ is a basis and $M = 0$. $\square$

Lemma 4.2.3 Assume that K is a finitely copresented module that is also formally smooth. Then it is finite free.

Proof Assume a finite copresentation:

$$0 \to K \to R^m \overset{M}{\to} R^n$$

We proceed by induction on m. By Lemma 4.2.2 we can decide whether $M = 0$ or not.

- If $M = 0$ then $K = R^m$ and we can conclude.
- If $M \neq 0$ then we can find a non-zero coefficient in the matrix corresponding to M, and so up to base change it is of the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \widetilde{M} & \\ 0 & & & \end{pmatrix}$$

  But then we know that the kernel of M is equivalent to the kernel of $\widetilde{M}$, and by applying the induction hypothesis we can conclude that it is finite free. □

Proposition 4.2.4 Let X be a smooth scheme. Then for any $x : X$ we have that $T_x(X)$ is finite free.

Proof By Lemma 4.2.1 we have that $T_x(X)$ is formally smooth, so that we can conclude by Lemma 4.2.3.□

The dimension of $T_x(X)$ is called the dimension of X at x. By boundedness any smooth scheme is a finite sum of smooth schemes of a fixed dimension. We can turn this into a definition of dimension which works well in the case of smooth schemes:

Definition 4.2.5 A scheme is *smooth of dimension* n, if it is smooth and all tangent spaces are finite free R-modules of dimension n.

4.3 Standard étale and standard smooth schemes

Definition 4.3.1 A standard smooth scheme of dimension k is an affine scheme of the form:

$$\mathrm{Spec}\left(R[X_1, \cdots, X_n, Y_1, \cdots Y_k]/P_1, \cdots, P_n\right)$$

where the determinant of:

$$\left(\frac{\partial P_i}{\partial X_j}\right)_{1 \leq i,j \leq n}$$

is invertible.

Definition 4.3.2 A standard smooth scheme of dimension 0 is called a *standard étale scheme*.

Lemma 4.3.3 Standard étale schemes are étale.

Proof Assume given a standard étale algebra:

$$R[X_1, \cdots, X_n]/P_1, \cdots, P_n$$

and write:

$$P : R^n \to R^n$$

for the map induced by $P_1, \cdots, P_n$.

Assume given $\epsilon : R$ such that $\epsilon^2 = 0$, we need to prove that there is a unique dotted lifting in:

$$\begin{tikzcd} R/\epsilon & R[X_1, \cdots, X_n]/P_1, \cdots, P_n \arrow[l, "x"'] \arrow[dl, dotted] \\ R \arrow[u] & \end{tikzcd}$$

This means that for all $x : R^n$ such that $P(x) = 0 \bmod \epsilon$, there exists a unique $y : R^n$ such that:
- We have $x = y \bmod \epsilon$.
- We have $P(y) = 0$.

First we prove existence. For any $b : R^n$ we compute:

$$P(x + \epsilon b) = P(x) + \epsilon\ dP_x(b)$$

We have that $P(x) = 0 \bmod \epsilon$, say $P(x) = \epsilon a$. Since $\neg\neg(P(x) = 0)$, we have that dP_x is invertible. Then taking $b = -(dP_x)^{-1}(a)$ gives a lift $y = x + \epsilon b$ such that $P(y) = 0$.

Now we check unicity. Assume y, y' two such lifts, then $y = y' \bmod \epsilon$ and we have:

$$P(y) = P(y') + dP_{y'}(y - y')$$

and $P(y) = 0$ and $P(y') = 0$ so that:

$$dP_{y'}(y - y') = 0$$

But $dP_{y'}$ is invertible and we can conclude that $y = y'$. □

Lemma 4.3.4 Any standard smooth scheme of dimension k is smooth of dimension k (Definition 4.2.5).

Proof The fibers of the map:

$$\mathrm{Spec}\left(R[X_1, \cdots, X_n, Y_1, \cdots Y_k]/P_1, \cdots, P_n\right) \to \mathrm{Spec}(R[Y_1, \cdots Y_k])$$

are standard étale, so the map is étale by Lemma 4.3.3. Since:

$$\mathrm{Spec}(R[Y_1, \cdots Y_k]) = \mathbb{A}^k$$

is smooth by Lemma 1.4.3, we can conclude it is smooth using Lemma 1.2.4.

For the dimension we use Lemma 2.2.3 and Corollary 4.1.2. □

4.4 Smooth schemes are locally standard smooth

Proposition 4.4.1 A scheme is smooth of dimension k if and only if it has a finite open cover by standard smooth schemes of dimension k.

Proof We can assume the scheme X affine, say of the form:

$$X = \mathrm{Spec}(R[X_1, \cdots, X_m]/P_1, \cdots, P_l)$$

By Proposition 4.2.4, for any $x : X$ we have that dP_x has free kernel. We partition by the dimension k of the kernel. Then by Lemma .0.5 we know that dP_x has rank $n = m - k$ for every x.

We cover X according to which n-minor is invertible, so that up to a rearranging of variables and polynomials we can assume that:

$$X = \mathrm{Spec}(R[X_1, \cdots, X_n, Y_1, \cdots, Y_k]/P_1, \cdots, P_n, Q_1, \cdots, Q_l)$$

where we have:

$$dP_{x,y} = \begin{pmatrix} \left(\frac{\partial P}{\partial X}\right)_{x,y} & \left(\frac{\partial P}{\partial Y}\right)_{x,y} \\ \left(\frac{\partial Q}{\partial X}\right)_{x,y} & \left(\frac{\partial Q}{\partial Y}\right)_{x,y} \end{pmatrix}$$

where we used the notation:

$$\left(\frac{\partial P}{\partial X}\right)_{x,y} = \left(\left(\frac{\partial P_i}{\partial X_j}\right)_{x,y}\right)_{i,j}$$

so that $\frac{\partial P}{\partial X}$ is invertible of size n. Moreover by Lemma .0.4 we get:

$$\left(\frac{\partial Q}{\partial Y}\right)_{x,y} = \left(\frac{\partial Q}{\partial X}\right)_{x,y}\left(\frac{\partial P}{\partial X}\right)^{-1}_{x,y}\left(\frac{\partial P}{\partial Y}\right)_{x,y}$$

which will be useful later.

Now we prove that for any $(x, y) : R^{n+k}$ such that $P(x, y) = 0$ it is decidable whether

$$Q(x, y) = 0$$

To do this it is enough to prove that:

$$(Q_1(x, y), \cdots, Q_l(x, y))^2 = 0 \to (Q_1(x, y), \cdots, Q_l(x, y)) = 0$$

Assuming $(Q_1(x, y), \cdots, Q_l(x, y))^2 = 0$, by smoothness there is a dotted lifting in:

$$\begin{tikzcd} R/(Q_1(x,y),\cdots,Q_l(x,y)) & \mathrm{Spec}(R[X_1,\cdots,X_n,Y_1,\cdots,Y_k]/P_1,\cdots,P_n,Q_1,\cdots,Q_l) \arrow[l, "{(x,y)}"'] \arrow[dl, dotted, "{(x',y')}"] \\ R \arrow[u] & \end{tikzcd}$$

Let us prove that $Q(x,y)=0$. Indeed we have $(x,y)\sim_1(x',y')$ so that we have:

$$P(x,y)=P(x',y')+\left(\frac{\partial P}{\partial X}\right)_{x',y'}(x-x')+\left(\frac{\partial P}{\partial Y}\right)_{x',y'}(y-y')$$

$$Q(x,y)=Q(x',y')+\left(\frac{\partial Q}{\partial X}\right)_{x',y'}(x-x')+\left(\frac{\partial Q}{\partial Y}\right)_{x',y'}(y-y')$$

Then we have $P(x,y)=0$, $P(x',y')=0$ and $Q(x',y')=0$. From the first equality we get:

$$x-x'=-\left(\frac{\partial P}{\partial X}\right)^{-1}_{x',y'}\left(\frac{\partial P}{\partial Y}\right)_{x',y'}(y-y')$$

so that from the second we get:

$$Q(x,y)=-\left(\frac{\partial Q}{\partial X}\right)_{x',y'}\left(\frac{\partial P}{\partial X}\right)^{-1}_{x',y'}\left(\frac{\partial P}{\partial Y}\right)_{x',y'}(y-y')+\left(\frac{\partial Q}{\partial Y}\right)_{x',y'}(y-y')$$

so that $Q(x,y)=0$ as we have seen previously that:

$$\left(\frac{\partial Q}{\partial Y}\right)_{x',y'}=\left(\frac{\partial Q}{\partial X}\right)_{x',y'}\left(\frac{\partial P}{\partial X}\right)^{-1}_{x',y'}\left(\frac{\partial P}{\partial Y}\right)_{x',y'}$$

From the decidability of $Q(x,y)=0$ we get that X is an open in $\mathrm{Spec}(R[X_1,\cdots,X_n,Y_1,\cdots,Y_k]/P_1,\cdots,P_n)$ so it is of the form $D(G_1,\cdots,G_n)$, and we have an open cover of our scheme by pieces of the form:

$$\mathrm{Spec}((R[X_1,\cdots,X_n,Y_1,\cdots,Y_k]/P_1,\cdots,P_n)_G)$$

Where $P_i(x)=0$ for all i and $G(x)\neq 0$ implies:

$$\det(\mathrm{Jac}(P_1,\cdots,P_n)_x)\neq 0$$

Then such a piece is equivalent to $\mathrm{Spec}\big(R[X_1,\cdots,X_n,X_{n+1},Y_1,\cdots,Y_k]/P_1,\cdots,P_n,GX_{n+1}-1\big)$ which is standard smooth as:

$$\det(\mathrm{Jac}(P_1,\cdots,P_n,GX_{n+1}-1))=\det(\mathrm{Jac}(P_1,\cdots,P_n)\cdot G$$

which is invertible. $\square$

Corollary 4.4.2 A scheme is formally étale if and only if it has a cover by standard étale schemes.

Proof By Corollary 3.1.2 we know that a scheme is formally étale if and only if it is smooth of dimension 0. Then we just apply Proposition 4.4.1. $\square$

Appendix

Rank of matrices

Definition .0.3 A matrix is said to have rank $\leq n$ if all its $n+1$-minors are zero. It is said to have rank n if it has rank $\leq n$ and does not have rank $\leq n-1$.

Having a rank is a property of matrices, as a rank function defined on all matrices would allow to e.g. decide if an $r : R$ is invertible.

Lemma .0.4 Assume given a matrix M of rank n decomposed into blocks:

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

Such that P is square of size n and invertible. Then we have:

$$S = RP^{-1}Q$$

Proof By columns manipulation the matrix is equivalent to:

$$M = \begin{pmatrix} P & Q \\ 0 & S - RP^{-1}Q \end{pmatrix}$$

but equivalent matrices have the same rank so $S = RP^{-1}Q$. □

Lemma .0.5 If a linear map $R^m \to R^n$ given by multiplication with M has finite free kernel of rank k, then M has rank $m-k$.

Proof Let $a_1, \ldots, a_k$ be a basis for the kernel of M in R^m, which we complete into a basis of R^m via $b_{k+1}, \ldots, b_m$. By completing $Mb_{k+1}, \ldots, Mb_m$ to a basis of R^n, we get a basis where M is written as:

$$\begin{pmatrix} I_{m-k} & 0 \\ 0 & 0 \end{pmatrix}$$

so that M has rank $m-k$. □

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