Châtelet's Theorem in Synthetic Algebraic Geometry

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Introduction

François Châtelet introduced the notion of Severi-Brauer variety in his 1944 PhD thesis [2]. One motivation is to provide a generalisation of the well-known result that a conic which has a rational point is isomorphic to $\mathbb{P}^1(k)$. He defines a Severi-Brauer variety to be a variety which becomes isomorphic to some $\mathbb{P}^n(k)$ after a separable extension. After recalling the characterisation of a central simple algebra over a field k, as an algebra which becomes isomorphic to a matrix algebra $M_n(k)$ after a separable extension, he notices the fundamental fact that, $\mathbb{P}^n(k)$ and $M_{n+1}(k)$ have the same automorphism group $PGL_{n+1}(k)$. He uses then this to describe a correspondence between Severi-Brauer varieties and central simple algebras, and as a corollary obtains the following generalisation of Poincaré's result: a Severi-Brauer variety which has a rational point is isomorphic to some $\mathbb{P}^n(k)$. This result and its proof are described in Serre's book on local fields [13]. (The paper [5] and the book [7] also contain a description of this result.)

The notion of central simple algebra over a field has been generalised to the notion of Azumaya algebra [1], and Grothendieck [8] generalized the notion of Severi-Brauer over over an arbitrary commutative ring. The goal of this note is to present a formulation and proof of Châtelet's Theorem over an arbitrary commutative ring in the setting of synthetic algebraic geometry [3], using the results already proved about projective space [4] in this context, in particular the fact that any automorphism of the projective space is given by a homography. We also rely essentially on basic results about dependent type theory with univalence [10] and modalities [12], in particular the fact that, in this context, étale sheafification can be described as modalities. The formulation of Châtelet's Theorem becomes that for X a scheme, we have that:

$$\|X = \mathbb{P}^n\|_T \to \|X\| \to \|X = \mathbb{P}^n\|$$

where:

$$\|X = \mathbb{P}^n\|_T$$

is the localisation of $\|X = \mathbb{P}^n\|$ for a modality T satisfying some basic properties (valid for the modality corresponding to étale sheafification).

1 Étale sheaves

1.1 Affine schemes are étale sheaves

Monic unramifiable polynomials are defined in [14] and analysed in [6]. If P is a proposition, we say that A is P-local if, and only if, the canonical map $A \to A^P$ is an equivalence. Given a family of propositions

P_i, the types that are P_i-local for all i form a model of type theory with univalence, and we have an associated lex modality, the nullification modality [12, 11]. Following [14], we can consider the étale modality, which corresponds to the family of propositions $\|\mathrm{Spec}(R[X]/g)\|$, for g monic unramifiable.

Definition 1.1. *A type X is called an étale sheaf if for all $g : R[X]$ monic unramifiable, we have that X is $\|\mathrm{Spec}(R[X]/g)\|$-local.*

Remark 1.2. *By [14] this should agree with the usual étale topology. It should also be noted that we will never use the unramifiability assumption, so we could just use non-constant monic polynomials instead.*

Lemma 1.1. *The type R is an étale sheaf.*

Proof. Let $g : R[X]$ be monic and write $S = \mathrm{Spec}(R[X]/g)$. We have a coequaliser in sets:

$$S \times S \rightrightarrows S \to \|S\|$$

So since R is a set we have an equaliser diagram:

$$R^{\|S\|} \to R^S \rightrightarrows R^{S\times S}$$

so that it is enough to prove that R is the equaliser of:

$$R[X]/g \rightrightarrows R[X]/g \otimes R[X]/g$$

to conclude. But since g is monic we merely have:

$$R[X]/g \simeq R^n$$

and it is clear that R is the equaliser of:

$$R^n \rightrightarrows R^n \otimes R^n$$

□

Remark 1.3. *If R is modal, then so is $\mathrm{Hom}(A, R)$ for any R-algebra A by general reasoning on modalities, so that every affine scheme is modal. By duality this implies that every finitely presented algebra is modal.*

1.2 Schemes are étale sheaves

Lemma 1.2. *Assume given a proposition P such that:*

- *The type R is P-local.*
- *Any open proposition is P-local.*
- *The type of open propositions is P-local.*

Then any scheme is P-local.

Proof. Since R is P-local, all affine schemes are P-local as explained in Theorem 1.3.

We check that for all scheme X, any map:

$$f : P \to X$$

merely factors through 1. Take $(U_i)_{i:I}$ a finite cover of X by affine scheme. Then for any $i : I$ we have that $f^{-1}(U_i)$ is an in P, so since the type of open is P-local, we merely have an open proposition V_i such that for all $x : P$, we have:

$$(x \in f^{-1}(U_i)) \leftrightarrow V_i$$

Since the $f^{-1}(U_i)$ cover P, we have that:

$$P \to \vee_{i:I} V_i$$

But open propositions are assumed to be P-local, so we have that:

$$\vee_{i:I} V_i$$

Assume $k : I$ such that V_k holds. Then $f^{-1}(U_k) = P$ and the map f factors through the affine scheme U_k. Since affine schemes are P-local, we merely have a lift for f.

Now we conclude that any scheme is P-local by proving that its identity types are P-local. Indeed they are schemes, so the previous point implies they are P-local. □

We will use freely the terminology and results of [3]; in particular a proposition is *open* if, and only if, it is equivalent to a proposition of the form $r_1 \neq 0 \vee \cdots \vee r_m \neq 0$ for some $r_1, \ldots, r_m$ in R.

Lemma 1.3. *If g is a monic polynomial, and $h_1, \ldots, h_m$ in $R[X]$, then the proposition $\forall_x g(x) = 0 \to h_1(x) \neq 0 \vee \cdots \vee h_m(x) \neq 0$ is open. It follows that, for any monic $g : R[X]$, and for any open U in $\mathrm{Spec}(R[X]/g)$ the proposition:*

$$\prod_{x:\mathrm{Spec}(R[X]/g)} U(x)$$

is open.

Proof. This follows from IV-10-2 in [9]. □

Proposition 1.1. *Any scheme is an étale sheaf.*

Proof. Assume given $g : R[X]$ monic, we can apply Lemma 1.2 because:

- The type R is an étale sheaf by Lemma 1.1.
- Any open proposition U is an étale sheaf because if:

  $$\|\mathrm{Spec}(R[X]/g)\| \to U$$

  then since $\neg\neg\,\mathrm{Spec}(R[X]/g)$ we have $\neg\neg U$, which implies U.
- Since open propositions are étale sheaves, it is enough that any map:

  $$\|\mathrm{Spec}(R[X]/g)\| \to \mathrm{Open}$$

  merely factors through 1. But given a constant open U in $\mathrm{Spec}(R[X]/g)$, for any $x : \mathrm{Spec}(R[X]/g)$ we have that:

  $$x \in U \leftrightarrow \prod_{y:\mathrm{Spec}(R[X]/g)} y \in U)$$

  The right hand side is open by Lemma 1.3, giving the required lift.

□

1.3 Descent for finite free modules

Lemma 1.4. *If we have M_x a finitely presented (resp. finite projective) R-module depending on $x : \mathrm{Spec}(A)$, then $\prod_{x:\mathrm{Spec}(A)} M_x$ is a finitely presented (resp. finite projective) A-module.*

Proof. See Theorem 7.2.3 in [3]. □

Lemma 1.5. *Let M be a module that is an étale sheaf such that we have the étale sheafification of "M is f.p.", then for any monic g we have that:*

$$R[X]/g \otimes M \simeq M^{\mathrm{Spec}(R[X]/g)}$$

Proof. We have that $R[X]/g \otimes M$ is merely equal to M^n where n is the degree of g, therefore it is an étale sheaf. As $M^{\mathrm{Spec}(R[X]/g)}$ is an étale sheaf as well, so when proving that:

$$R[X]/g \otimes M \to M^{\mathrm{Spec}(R[X]/g)}$$

is an equivalence, we can assume that M is finitely presented. In this case we conclude by Theorem 7.2.3 in [3]. $\square$

Lemma 1.6. *Let A be an fppf algebra and let M be an R-module. Then if $A \otimes M$ is f.p. (resp. finite projective) as an A-module if and only if M is f.p. (resp. finite projective) as an R-module.*

Proof. See VIII.6.7 in [9]. $\square$

Proposition 1.2. *For M a module that is an étale sheaf, the proposition "M is an finite free" is itself an étale sheaf.*

Proof. Follows from Lemmas 1.5 and 1.6. $\square$

2 Azumaya algebras and their associated Severi-Brauer variety

From now on we assume a lex modality T such that:

- Schemes are modal.
- If M is a T-modal R-module, then the proposition of M being finite free is modal.

We call T-modal types sheaves and we write $\|X\|_T$ the sheafification of the propositional truncation of X. Note that T-modal types form a model of homotopy type theory [12, 11].

In Section 1 we constructed such a modality (Proposition 1.1 and Proposition 1.2).

We fix a natural number n throughout.

2.1 The type AZ_n of Azumaya algebras

Definition 2.1. *An Azumaya algebra of rank n is a (non-commutative, unital) R-algebra A such that its underlying type is a sheaf and:*

$$\|A = M_{n+1}(R)\|_T$$

We write AZ_n for the type of Azumaya algebra of rank n.

Remark 2.2. *In [6], we give a constructive proof that a R-algebra A is an Azumaya algebra of rank n if, and only if, A is free as a R-module of rank $(n+1)^2$ and the canonical map $A \otimes A^{op} \to \mathrm{End}_R(A)$ is an isomorphism.*

Lemma 2.1. *For all $A : \mathrm{AZ}_n$ we have that A is finite free as a module.*

Proof. By hypothesis A being finite free is modal so that $\|A = M_{n+1}(R)\|_T$ implies A finite free. $\square$

Definition 2.3. *Let V be a free R-module, we define $\mathrm{Gr}_k(V)$ the k-Grassmannian of V as the type of k-dimensional subspaces of V.*

Lemma 2.2. *Let V be a finite free module, then $\mathrm{Gr}_k(V)$ is a scheme.*

Proof. We can assume $V = R^n$. The type of k-dimensional subspaces of R^n is the type of $n \times k$ matrices of rank k quotiented by the natural action of GL_k. For all $k \times k$ minor, we consider the open proposition stating this minor is non-zero, which well defined as it is invariant under the GL_k-action. This gives a finite open cover of $\mathrm{Gr}_k(R^n)$.

Let us show any piece is affine. For example consider the piece of matrices of the form:

$$\begin{pmatrix} P & N \end{pmatrix}$$

where P is invertible of size $k \times k$. Any orbit in this piece has a unique element of the form:

$$\begin{pmatrix} I_k & N' \end{pmatrix}$$

where I_k is the identity matrix, so this piece is equivalent to $R^{(n-k)k}$. $\square$

Lemma 2.3. *For all $A : \mathrm{AZ}_n$ and $I : \mathrm{Gr}_{n+1}(A)$, we have that I being a right ideal in A is a closed proposition.*

Proof. By Lemma 2.1 we have that A is finite free as a module. Consider $a_0, \cdots, a_n$ a basis of I and extend it to a basis of A adding $b_1, \cdots, b_l$. We can proceed as if R was a field because a non zero vector has a non-zero coefficient [3].

For any $a : A$, we have that $a \in I$ is a closed proposition as it says that the $b_1, \cdots, b_l$ coordinates of a are zero.

Then I is an ideal if and only if for any a in the chosen basis of A and any i in the chosen basis of I we have that $ai \in I$, which is a closed proposition. □

Lemma 2.4. *For all $A : \mathrm{AZ}_n$ we define:*

$$\mathrm{RI}(A) := \{I : \mathrm{Gr}_{n+1}(A) \mid I \text{ is a right ideal}\}$$

Then $\mathrm{RI}(A)$ is a scheme.

Proof. By Lemma 2.1 we have that A is finite free as a module, so that by Lemma 2.2 we have that $\mathrm{Gr}_{n+1}(A)$ is a scheme, and then by Lemma 2.3 we have that $\mathrm{RI}(A)$ is closed in a scheme, so it is a scheme. □

2.2 Quaternion algebras are Azumaya algebras

In this section we assume $2 \neq 0$, and we take T to be the étale sheafification.

Definition 2.4. *Given $a, b : R^\times$, we define the quaternion algebra $Q(a, b)$ as the non-commutative algebra:*

$$R[i, j]/(i^2 = a, j^2 = b, ij = -ji)$$

Remark 2.5. *As a vector space, $Q(a, b)$ is of dimension 4, generated by $1, i, j, ij$.*

Remark 2.6. *By the change of variable $i \mapsto j$ and $j \mapsto i$ we get $Q(a, b) = Q(b, a)$.*

Lemma 2.5. *For all $b : R^\times$, we have that $Q(1, b) = M_2(R)$.*

Proof. We send i to:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and j to:

$$J = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

Then IJ is:

$$K = \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}$$

It is easy to check this define an algebra morphism, and since $1, I, J, K$ form a basis of $M_2(R)$ the map is an isomorphism. □

Lemma 2.6. *For all $a, b, u, v : R^\times$, we have that $Q(a, b) = Q(u^2 a, v^2 b)$.*

Proof. We use the variable change $i \mapsto ui$ and $j \mapsto vj$. □

Lemma 2.7. *Given $a, b : R^\times$, we have that $Q(a, b)$ is an Azumaya algebra.*

Proof. We have that $Q(a, b)$ is finite free as a vector space so it is a sheaf. So $Q(a, b)$ being Azumaya is a sheaf and we can assume $\sqrt{a}$. Then by Lemma 2.6 we have $Q(1, b) = Q(\sqrt{a}^2, b) = Q(a, b)$ and we conclude by Lemma 2.5. □

2.3 A remark on Azumaya algebras

Lemma 2.8. *For any $n : \mathbb{N}$, the map:*

$$M_{n+1}(R) \otimes M_{n+1}(R)^{op} \to \mathrm{End}_R(M_{n+1}(R))$$

$$M \otimes N \mapsto (P \mapsto MPN)$$

is an equivalence.

Proof. Let us denote by $(E_{i,j})_{0 \leq i,j \leq n}$ the canonical basis of $M_{n+1}(R)$. We consider the basis:

$$(E_{i,j} \otimes E_{k,l})_{0 \leq i,j,k,l \leq n}$$

of $M_{n+1}(R) \otimes M_{n+1}(R)^{op}$, as well as the basis:

$$(C_{i,j,k,l})_{0 \leq i,j,k,l \leq n}$$

of $\mathrm{End}_R(M_{n+1}(R))$ where $C_{i,j,k,l}(E_{j,k}) = E_{i,l}$ and $C_{i,j,k,l}$ is null on other element of the basis. It is clear that the morphism sends one basis to the other, and that both algebras have the same multiplication table. □

Lemma 2.9. *Assume $A : \mathrm{AZ}_n$, then A is finite free as a module and the map $A \otimes A^{op} \to \mathrm{End}_R(A)$ sending $a \otimes b$ to $c \mapsto acb$ is an equivalence.*

Proof. The fact that A is finite free is Lemma 2.1. Then both $A \otimes A^{op}$ and $\mathrm{End}_R(A)$ are finite free modules and therefore are T-modal, so that the map being an equivalence is T-modal and when proving it we can assume $A = M_{n+1}(R)$. Then we conclude by Lemma 2.8. □

2.4 The type SB_n of Severi-Brauer varieties

Definition 2.7. *A type X is called a Severi-Brauer variety of dimension n if X is a sheaf and:*

$$\|X = \mathbb{P}^n\|_T$$

We write SB_n the type of Severi-Brauer varieties of dimension n. We will see later that every Severi-Brauer variety is a scheme.

Lemma 2.10. *Consider the map:*

$$\delta : \mathbb{P}^n \to \mathrm{RI}(M_{n+1}(R))$$

sending $(x_0 : \cdots : x_n) : \mathbb{P}^n$ to:

$$\{M : M_n(R) \mid \forall i,j.\ x_i \cdot M_j = x_j \cdot M_i\}$$

where M_i is the i-th line of M. Then δ is an equivalence.

Proof. Write $X = (x_0 : \cdots : x_n)$. First we check δ is well defined. It is clear that for all $\lambda \neq 0$ we have that:

$$\delta(\lambda X) = \delta(X)$$

and that $\delta(X)$ is a right ideal. To check the dimension assume $x_k \neq 0$. Then $M \in \delta(X)$ if and only if for all i we have that $M_i = \frac{x_i}{x_k} M_k$, which means giving $M \in \delta(X)$ is equivalent to giving M_k in R^{n+1}, so $\delta(X)$ is free of dimension $n+1$.

Next we check injectivity. Assume given $(x_0 : \cdots : x_n)$ and $(y_0 : \cdots : y_n)$ in $\mathbb{P}^n$ such that for all $M : M_n(R)$ we have:

$$(\forall i,j.\ x_i \cdot M_j = x_j \cdot M_i) \leftrightarrow (\forall i,j.\ y_i \cdot M_j = y_j \cdot M_i)$$

In particular considering the matrix N such that $N_j = (y_j, \cdots, y_j)$ we get that:

$$\forall i,j.\ x_i y_j = x_j y_i$$

so that:

$$(x_0 : \cdots : x_n) = (y_0 : \cdots : y_n)$$

Finally we check surjectivity. Assume $I : \mathrm{RI}(M_{n+1}(R))$, since $\|I = R^{n+1}\|$ we have $M \in I$ such that $M \neq 0$, for example assume $M_{0,0} \neq 0$. Then for all k we have that:

$$ME_{0,k} \in I$$

meaning that we have $N^k \in I$ where:

$$N^k_{i,k} = M_{i,0}$$

and when when $j \neq k$

$$N^k_{i,j} = 0$$

Then since $M_{0,0} \neq 0$ the matrices N^k are linearly independent and since I has dimension $n+1$, it is precisely the ideal spanned by the N^k. But this ideal is $\delta(M_{0,0} : \cdots : M_{n,0})$. □

Lemma 2.11. *If A is an Azumaya algebra, then* $\mathrm{RI}(A)$ *is a Severi-Brauer variety.*

Proof. By Lemma 2.4 and the assumption that schemes are sheaves, we have that $\mathrm{RI}(A)$ is a sheaf. Then to prove:

$$\|A = M_{n+1}(R)\|_T \to \|\mathrm{RI}(A) = \mathbb{P}^n\|_T$$

it is enough to prove:

$$\mathrm{RI}(M_{n+1}(R)) = \mathbb{P}^n$$

which is Lemma 2.10. □

2.5 Conics are Severi-Brauer varieties

In this section we assume $2 \neq 0$, and we take T to be the étale sheafification.

Definition 2.8. *Given $a, b : R^\times$, we define the conic $C(a,b)$ as the set of $[x : y : z] : \mathbb{P}^2$ such that $x^2 = ay^2 + bz^2$.*

Lemma 2.12. *Assume $a, b : R^\times$ such that $\|C(a,b)\|$, then $\|C(a,b) = \mathbb{P}^1\|$.*

Proof. Let us assume x_0, y_0, z_0 such that $x_0^2 = ay_0^2 + bz_0^2$. We can assume $x_0 \neq 0$ by possibly considering $C(a,b) = C(\frac{1}{a}, -\frac{b}{a})$. Then we can clearly assume $x_0 = 1$ without loss of generality, so that $ay_0 + bz_0 = 1$.

Let us consider the map:

$$\psi : \mathbb{P}^1 \to \mathbb{P}^2$$

$$[u : v] \mapsto [au^2 + bv^2 : y_0(au^2 - bv^2) + 2buvz_0 : z_0(au^2 - bv^2) - 2auvy_0]$$

We want to define ϕ inverse to ϕ. Assume $[x : y : z] : \mathbb{P}^2$ such that $x^2 = ay^2 + bz^2$.

Let us proof that either $x + ay_0y + bz_0z$ or $x - ay_0y - bz_0z$ is invertible, to do this it is enough to prove that either x or $ay_0y + bz_0z$ is invertible. Assume $x = 0$ and $ay_0y + bz_0z = 0$, then we have a contradiction. Indeed y or z is invertible and $ay^2 + bz^2 = 0$, so that y and z are invertible and $b = -a\frac{y^2}{z^2}$ and $y_0z = yz_0$. Moreover y_0 or z_0 is invertible, so that both y_0 and z_0 are invertible and $\frac{y}{z} = \frac{y_0}{z_0}$ which means $ay^2 + bz^2 = 0$ implies $ay_0^2 + bz_0^2 = 0$, a contradiction.

If $x + ay_0y + bz_0z$ is invertible we define $\phi([x, y, z]) = [1 : \frac{a(z_0y - y_0z)}{x + ay_0y + bz_0z}]$.

If $x - ay_0y - bz_0z$ is invertible we define $\phi([x, y, z]) = [\frac{b(z_0y - y_0z)}{x - ay_0y - bz_0z} : 1]$.

This is well defined as if both are invertible then:

$$\frac{b(z_0y - y_0z)}{x - ay_0y - bz_0z} \times \frac{a(z_0y - y_0z)}{x + ay_0y + bz_0z} = \frac{ab(z_0y - y_0z)^2}{x^2 - (ay_0y + bz_0z)^2} = 1$$

because:

$$x^2 - (ay_0y + bz_0z)^2 = (ay^2 + bz^2)(ay_0^2 + bz_0^2) - (ay_0y + bz_0z)^2 = ab(z_0y - y_0z)^2$$

We omit the verification that this ϕ is indeed an inverse to ψ. □

Lemma 2.13. *Assume given $b : R^\times$, then $\|C(1,b) = \mathbb{P}^1\|$.*

Proof. We just apply Lemma 2.12 to the point $[1,1,0] : C(1,b)$. □

Lemma 2.14. *Given $a,b,u,v : R^\times$ we have that $C(a,b) = C(u^2a, v^2b)$.*

Proof. Consider the change of variable $y \mapsto uy$ and $z \mapsto vz$. □

Lemma 2.15. *Assume given $a,b : R^\times$, then $C(a,b)$ is a Severi-Brauer variety.*

Proof. Since a sheaf being a Severi-Brauer variety is an étale sheaf, we can assume $\sqrt{a}$. Then by Lemma 2.14 we have that $C(1,b) = C(\sqrt{a}^2, b) = C(a,b)$ and we conclude by Lemma 2.13. □

Remark 2.9. *We will see in Theorem 3.4 that any merely inhabited Severi-Brauer variety is a projective space.*

3 The equivalence $\mathrm{AZ}_n \simeq \mathrm{SB}_n$ and Châtelet's Theorem

3.1 Generalities on delooping in T-sheaves

Definition 3.1. *A type A is T-connected if:*

$$\forall (x,y : A).\ \|x = y\|_T$$

The key intuition for the next lemma is that both A and B are deloopings of the same group in the topos of sheaves.

Lemma 3.1. *Assume A, B pointed T-connected sheaves. Let $f : A \to B$ be a pointed map inducing an equivalence:*

$$\Omega f : \Omega A \simeq \Omega B$$

Then f is an equivalence.

Proof. First we prove that f is an embedding. We have to prove that for all $x, y : A$ the map:

$$\mathrm{ap}_f : x = y \to f(x) = f(y)$$

is an equivalence. Since A and B are sheaves so are their identity types, so ap_f being an equivalence is a sheaf, so by T-connectedness of A we can assume x and y are the basepoint, in which case it is part of the hypothesis.

Now we prove it is surjective, indeed for any $x : B$ we have that $\mathrm{fib}_f(x)$ is a sheaf and a proposition so when proving it is inhabited we can assume x is the basepoint of B and give the basepoint of A as antecedent. □

3.2 Both $\mathrm{Aut}(M_{n+1}(R))$ and $\mathrm{Aut}(\mathbb{P}^n)$ are $\mathrm{PGL}_{n+1}(R)$

We need an well-known algebra result.

Lemma 3.2. *Let M be a finite projective module, then M is finite free.*

Proof. This is IX.2.2 in [9], it crucially relies on R being local. □

Lemma 3.3. *Assume $e_{i,j} : M_{n+1}(R)$ for $0 \le i,j \le n$ such that:*

$$e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$$

where $\delta_{j,k} = 1$ if $j = k$ and 0 otherwise. Moreover assume:

$$e_{0,0} + \cdots + e_{n,n} = 1$$

Then there exists $P : GL_{n+1}(R)$ such that:

$$e_{i,j} = PE_{i,j}P^{-1}$$

Proof. We define $e_i = e_{i,i}$, then $e_0 + \cdots + e_n = 1$, for all i we have $e_i^2 = e_i$ and for all $i \neq j$ we have that $e_i e_j = 0$. From this we get:

$$R^{n+1} = V_0 \oplus \cdots \oplus V_n$$

where:

$$V_i = \{x \mid e_i(x) = x\}$$

and:

$$e_{i,j} : V_j \simeq V_i$$

As a direct summand of a free module we have that V_0 is projective, and since $V_0 = e_0(R^{n+1})$ we have that V_0 is finitely generated, so by Lemma 3.2 it is finite free. From $V_0^{n+1} = R^{n+1}$ we get that $\|V_0 = R\|$, and therefore that $\|V_i = R\|$ for all i.

Then we choose v_0 generating V_0 and define $v_i = e_{i,0}(v_0)$ so that v_i generates V_i because $e_{i,0} : V_0 \simeq V_i$. We get a basis $v_0, \cdots, v_n$ of R^{n+1}.

Let $u_0, \cdots, u_n$ be the canonical basis of R^{n+1} and define $P : GL_{n+1}(R)$ by sending u_i to v_i. Then for all v_k we have that:

$$e_{i,j} v_k = P E_{i,j} P^{-1} v_k$$

so we can conclude. $\square$

Proposition 3.1. *The map:*

$$\alpha : \mathrm{PGL}_{n+1}(R) \to \mathrm{Aut}(M_{n+1}(R))$$
$$P \mapsto (M \mapsto PMP^{-1})$$

is an equivalence.

Proof. It is clearly a group morphism.

For injectivity we just need to check that if for all $M : M_{n+1}(R)$ we have $PMP^{-1} = M$ then there exists $\lambda \neq 0$ such that $P = \lambda I_{n+1}$. We deduce this from $Pe_{i,j} = e_{i,j}P$ and P invertible.

For surjectivity consider $e_{i,j} = \sigma(E_{i,j})$, we can apply Lemma 3.3 to get $P : GL_{n+1}(R)$ such that:

$$\sigma(E_{i,j}) = P E_{i,j} P^{-1}$$

from which we conclude that for all $M : M_{n+1}(R)$ we have that:

$$\sigma(M) = PMP^{-1}$$

as desired. $\square$

Proposition 3.2. *The map:*

$$\beta : \mathrm{PGL}_{n+1}(R) \to \mathrm{Aut}(\mathbb{P}^n)$$
$$X \mapsto PX$$

is an equivalence.

Proof. This is the main result from [4]. $\square$

3.3 The Severi-Brauer construction is an equivalence

Proposition 3.3. *The map:*

$$\mathrm{RI} : \mathrm{AZ}_n \to \mathrm{SB}_n$$

is an equivalence.

Proof. By Lemma 3.1 it is enough to prove that the top map in the triangle:

$$\begin{tikzcd}
\mathrm{Aut}(M_n(R)) \arrow[rr, "\Omega\mathrm{RI}"] & & \mathrm{Aut}(\mathbb{P}^n) \\
& \mathrm{PGL}_{n+1}(R) \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

is an equivalence. But since the two other maps in the triangle are equivalences by Proposition 3.1 and Proposition 3.2, it is enough to prove that the triangle commutes. To do this we need to check that for all $P : \mathrm{PGL}_{n+1}(R)$ we have that:

$$\delta^{-1} \circ \mathrm{ap}_{\mathrm{RI}}(\alpha(P)) \circ \delta = \beta(P)$$

in $\mathrm{Aut}(\mathbb{P}^n)$, with δ defined in Lemma 2.10. So we need to prove the following square commutes:

$$\begin{tikzcd}
\mathrm{RI}(M_{n+1}(R)) \arrow[r, "I \mapsto PIP^{-1}"] & \mathrm{RI}(M_{n+1}(R)) \\
\mathbb{P}^n \arrow[u, "\delta"] \arrow[r, "X \mapsto PX"'] & \mathbb{P}^n \arrow[u, "\delta"']
\end{tikzcd}$$

where $\mathrm{ap}_{\mathrm{RI}}$ was computed using path induction.

We see that:

$$\delta(Y) = \{M : M_n(R) \mid \forall A, B : R^{n+1}.\ A^t X \cdot B^t M = B^t X \cdot A^t M\}$$

To check that:

$$P\delta(X)P^{-1} = \delta(PX)$$

we just need to check an inclusion as both are finite free modules of the same dimension. Assume $M \in \delta(X)$, to check that $PMP^{-1} \in \delta(PX)$ we need to check that for all $A, B : R^{n+1}$ we have that:

$$A^t PX \cdot B^t PMP^{-1} = B^t X \cdot A^t PMP^{-1}$$

but since $M \in \delta(X)$ we have that:

$$(P^t A)^t X \cdot (P^t B)^t M = (P^t B)^t X \cdot (P^t A)^t M$$

which gives us what we want. □

Remark 3.2. *By Lemma 2.4 and Proposition 3.3 we can conclude than any Severi-Brauer variety is a scheme. This was not clear a priori because being a scheme is not modal.*

Remark 3.3. *By Proposition 3.3 and Theorem 2.2 and we can conclude that a type X being a Severi-Brauer variety for any modality T such that:*

- *Schemes are T-modal.*
- *The type of finite free modules is T-modal.*
- *T-modal types are étale sheaves.*

is equivalent to X being a Severi-Brauer variety for the étale topology. In particular Severi-Brauer varieties do not depend on the choice of such a T.

3.4 Châtelet's Theorem

Lemma 3.4. *Assume $A : \mathrm{AZ}_n$ with $I : \mathrm{RI}(A)$, then:*

$$A = \mathrm{End}_R(I)^{op}$$

Proof. Since I is an ideal, there is a canonical map of algebra:

$$A \to \mathrm{End}_R(I)^{op}$$

Since both algebras are sheaves (indeed $\|I = R^{n+1}\|$ implies I is a sheaf), this map being an equivalence is a sheaf so we can assume $A = M_{n+1}(R)$.

By Lemma 2.10 we can assume $X = (x_0 : \cdots : x_n) : \mathbb{P}^n$ such that $I = \delta(X)$. There is a k such that $x_k \neq 0$, so we can assume $x_k = 1$, then we have an isomorphism:

$$\theta : R^{n+1} \to I$$

sending $Y : R^{n+1}$ to the matrix M with its i-th line $M_i = x_i Y$. Then for all $M : M_n(R)$ we have a commutative square:

$$\begin{tikzcd} I \arrow[r, "N \mapsto NM"] & I \\ R^{n+1} \arrow[u, "\theta"] \arrow[r, "X \mapsto M^t X"'] & R^{n+1} \arrow[u, "\theta"'] \end{tikzcd}$$

meaning the natural map:

$$M_{n+1}(R) \to \mathrm{End}_R(I)^{op}$$

sends M to $\delta^{-1} \circ M \circ \delta$, so it is an equivalence. □

Theorem 3.4 (Châtelet's Theorem)**.** *Assume $X : \mathrm{SB}_n$, then:*

$$\|X\| \to \|X = \mathbb{P}^n\|$$

Proof. By Proposition 3.3 we can assume $X = \mathrm{RI}(A)$ for some $A : \mathrm{AZ}_n$. Then we can assume $I : \mathrm{RI}(A)$, so that by Lemma 3.4 we have that:

$$A = \mathrm{End}_R(I)^{op}$$

Since we merely have that $I = R^{n+1}$, we merely have:

$$A = M_{n+1}(R)^{op} = M_{n+1}(R)$$

Applying Lemma 2.10 we merely conclude:

$$X = \mathrm{RI}(A) = \mathrm{RI}(M_{n+1}(R)) = \mathbb{P}^n$$

□

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