Quasi-categories and Complete Segal Spaces

Hugo Moeneclaey,
hmoenecl@ens-paris-saclay.fr

Under the supervision of :

Georges Maltsiniotis,
Institut de Mathématiques de Jussieu
georges.maltsiniotis@imj-prg.fr

November 8, 2018

Abstract

We present an article by Joyal and Tierney [8]. It builds two Quillen equivalences between the Joyal model structure on simplicial sets and the Rezk model structure on bisimplicial sets.

Contents

1 Introduction

This document presents an article by Joyal and Tierney [8]. It is about some formalisations of the notion of $(\infty, 1)$-category. We now give some intuitions about this notion.

First we introduce ∞-groupoids. It is possible to associate to a space the collections of its points, paths, homotopies between paths, homotopies between these homotopies, and so on. These collections inherit a lot of structure from the space. The notion of ∞-groupoids should axiomatize this structure. Therefore an ∞-groupoid has some objects, some morphisms between objects, some morphisms between morphisms between objects, and so on. It also has identities and some composition laws for morphisms. The correspondence between these categorical and topological concepts is summarised in the following figure.

Topological space	∞-groupoid
point	object
path	morphism
homotopy	morphism between morphisms
constant path or homotopy	identity morphism
concatenation of paths or homotopies	compostion law

Since paths and homotopies can be travelled backward, it is required that morphisms are invertible up to higher morphisms, i.e. for any morphism $\alpha : f \to g$ there exists a morphism $\beta : g \to f$ together with higher morphisms $h_1 : \mathrm{id}_g \to \alpha \circ \beta$ and $h_2 : \beta \circ \alpha \to \mathrm{id}_f$. In this case α is said weakly invertible.

Moreover it is required that some rules hold up to coherent homotopy. It is this notion of coherent homotopy which is hard to formalise. For example an ∞-groupoid should have some weakly invertible morphisms $\alpha_{f,g,h} : (f \circ g) \circ h \to f \circ (g \circ h)$ for any composable morphisms f, g and h. Moreover these morphisms should obey to the pentagon law, i.e. the following diagram should commute up to a higher morphism :

$$\begin{tikzcd}
& ((f \circ g) \circ h) \circ i \arrow[dl, "\alpha_{f \circ g, h, i}"'] \arrow[dr, "\alpha_{f,g,h} \circ i"] & \\
(f \circ g) \circ (h \circ i) \arrow[d, "\alpha_{f,g,h \circ i}"'] & & (f \circ (g \circ h)) \circ i \arrow[d, "\alpha_{f, g \circ h, i}"] \\
f \circ (g \circ (h \circ i)) & & f \circ ((g \circ h) \circ i) \arrow[ll, "f \circ \alpha_{g,h,i}"]
\end{tikzcd}$$

But these higher morphisms should themselves have some kind coherence morphisms, and so on.

Then $(\infty, 1)$-categories are the generalisation of ∞-groupoids where we do not require morphisms between objects to be weakly invertible. Note that we still require morphisms between morphisms to be weakly invertible. But how to formalise this notion of $(\infty, 1)$-categories ?

A natural first step is to formalise the notion of ∞-groupoids. This can be done using a guiding principle in the study of $(\infty, 1)$-categories called the homotopy hypothesis, which states that ∞-groupoids are equivalent to weak homotopy types.

This motivates a fundamental insight : the notion of $(\infty, 1)$-category is a homotopical notion of some sort, and similarly to weak homotopy types it is best formalised as a category $\mathcal{C}$ and a class of morphisms W in $\mathcal{C}$. The idea is that the homotopy category of $(\infty, 1)$-categories is the category $\mathcal{C}$ localised at W (i.e. the category obtained from $\mathcal{C}$ by formally inverting the morphisms in W). At this point it should be noted that the homotopy category of $(\infty, 1)$-categories is badly behaved (as is the homotopy category of weak homotopy types), and that it is often more useful to consider the $(\infty, 1)$-category of $(\infty, 1)$-categories. It is also useful to consider the $(\infty, 1)$-category of weak homotopy types, in fact this is is the main example motivating $(\infty, 1)$-categories.

The notion of model category [11] is an additional structure on a category $\mathcal{C}$ together with a class of morphisms W called the weak equivalences. It allows to effectively manipulate it, for example to compute the localisation of $\mathcal{C}$ at W. The prime example is the Quillen model structure on simplicial sets, which represents weak homotopy types (and therefore ∞-groupoids by the homotopy hypothesis). There is a lot of different formalisations of $(\infty, 1)$-categories as a model category in the literature. We list some of these model structures :

- Perhaps the most intuitive is the model structure on simplicially enriched categories considered in [2]. The point of view of this structure is that an $(\infty, 1)$-category is a category enriched in ∞-groupoids. Note that in this model the associativity of the composition of morphisms in an $(\infty, 1)$-category holds on the nose, so this model structure can be interpreted as a strictification result.

- The most well-known is called the Joyal model structure on the category of simplicial sets. It emphasises the similarity between $(\infty, 1)$-categories and ∞-groupoids. Results about this model structure are recalled without proof in Section 2.3.

- There is such a model structure on bisimplicial sets first defined in [12], which we call the Rezk model structure. This model structure is presented in Section 4.5.

- Another model structure which is enlightening is the model structure on relative categories presented in [1]. A relative category is a category $\mathcal{C}$ together with a class of morphism W containing the identities and stable by composition. The idea is that a relative category $(\mathcal{C}, W)$ represents the $(\infty, 1)$-category obtained from the degenerate $(\infty, 1)$-category $\mathcal{C}$ by formally inverting W in the world of $(\infty, 1)$-categories. This process is called simplicial localisation. It can be used to define the $(\infty, 1)$-category of $(\infty, 1)$-categories from any model structure representing $(\infty, 1)$-categories.

All these model categories represent the same notion of $(\infty, 1)$-category, so for example they have equivalent localisations and equivalent simplicial localisations. This is implied by the fact that they are all linked by chains of special morphisms called Quillen equivalences. The main goal of this document is to present two Quillen equivalences between the Joyal and Rezk model structures.

In any model category we have a notion of fibrant objects, which are well-behaved objects. We will now give an interpretation of fibrant objects in the Joyal and Rezk model structure as $(\infty, 1)$-categories. In order to interpret an arbitrary (bi)simplicial set X as an $(\infty, 1)$-category, one should first choose a fibrant replacement for X, i.e. a fibrant object weakly equivalent to X. The fibrant objects in the Joyal model structure are called quasi-categories, and the fibrant objects in the Rezk model structure are called complete Segal spaces.

A notion which is important in both model structures is the notion of commutative n-simplex in an $(\infty, 1)$-category. A commutative 2-simplex (also called a commutative triangle) is the data of morphisms $f : x \to y$, $g : y \to z$ and $h : x \to z$ together with a higher morphism $g \circ f \to h$. The commutative n-simplices are defined similarly with edges being morphims between objects, surfaces being morphisms between morphisms between objects, and so on.

The Joyal model structure represents $(\infty, 1)$-categories as nice simplicial sets called quasi-categories. We give some intuitions about this correspondence. For X a quasi-category, the set X_0 represents objects of the $(\infty, 1)$-category corresponding to X, the set X_1 represents its morphisms (with the face maps indicating sources and targets). Then X_n for $n \geq 2$ represents its commutative n-simplices.

The Rezk model structure is a model structure on bisimplicial sets. To a bisimplicial set X we can associate its rows denoted by $X_{m,\bullet}$ and its columns denoted by $X_{\bullet,n}$. They are simplicial sets. Let us denote by $\mathcal{C}$ the $(\infty, 1)$-category corresponding to a complete Segal space X. Then $X_{\bullet,0}$ represents the space of objects of $\mathcal{C}$, which has the objects of $\mathcal{C}$ as points, the weakly invertible morphisms of $\mathcal{C}$ as paths and higher weakly invertible morphisms as homotopies. Moreover $X_{\bullet,1}$ is the space of morphisms of $\mathcal{C}$, which has the morphisms of $\mathcal{C}$ as points, weakly invertible morphisms between them as paths (i.e commutative square with weakly invertible edges), and higher weakly invertible morphisms as homotopies. Similarly $X_{\bullet,n}$ for $n \geq 2$ is the space of commutative n-simplices in $\mathcal{C}$. Then $X_{\bullet,n}$ for $n \geq 2$ is homotopically equivalent to the space of strings of n composable morphisms in $\mathcal{C}$, since there is a unique way up to homotopy to fill a string of n morphisms into a commutative n-simplex.

The first Quillen equivalence should be clear from this presentation : to a bisimplicial set we associate its row $X_{0,\bullet}$, which has the points of $X_{\bullet,0}$ as objects, the points of $X_{\bullet,1}$ as morphisms, and so on.

We now present the second Quillen equivalence. We denote by Δ^n the standard n-simplex, and by $(\Delta^n)'$ the nerve of the groupoid with $n+1$ canonically isomorphic objects. Let X be a simplicial set representing the $(\infty, 1)$-category $\mathcal{C}$, then the second Quillen equivalence associates to it the bisimplicial set Y defined by $Y_{\bullet,m} = \mathrm{Hom}_{\mathbf{S}}(\Delta^m \times (\Delta^\bullet)', X)$. This is indeed a satisfying representation of the spaces of objects, morphisms and commutative m-simplices in $\mathcal{C}$.

We will now present the structure of the proof. The weak equivalences in the Quillen (resp. Joyal) model structure are called the weak homotopy equivalences (resp. weak categorical equivalences).

First we build a model structure on bisimplicial sets called the vertical model structure, where the weak equivalences are the column-wise weak homotopy equivalences. Then we present the general theory of Bousfield localisations allowing us to build new model structures on bisimplicial sets with more weak equivalences than the vertical model structure. We use this to define the Rezk model structure, in which the intuitions presented earlier are true for fibrant objects. Then we check that row-wise weak categorical equivalences are Rezk weak equivalences. From this result it is possible to construct the first Quillen equivalence, and prove that it is indeed an equivalence by using carefully chosen fibrant replacements in the Rezk model structure. Then we construct the second Quillen equivalence and we prove that it is indeed an equivalence using the first one and the two-out-of-three property of Quillen equivalences.

2 Model structures on simplicial sets

We denote by $\mathbf{S}$ the category of simplicial sets. In this section we present two model structures on $\mathbf{S}$. We call Quillen model structure the structure modelling weak homotopy types, and we call Joyal model structure the structure modelling $(\infty, 1)$-categories.

The reader should have a look at Appendix A.1 at this point, it presents the notions of left and right lifting properties, and the notion of saturated class of morphisms. He should also have a look at Appendix B.1 which presents model categories, and at Appendices B.2 and B.3 which present the notions of Quillen functors and Quillen equivalences.

There is almost no proof in this section, the proofs can be found for example in [7].

2.1 Some notations for maps of simplicial sets

We denote by δ^n the inclusion map $\partial\Delta^n \subset \Delta^n$ for $n \geq 0$.

We denote by h^n_k the inclusion map of the k-th horn $\Lambda^n_k \subset \Delta^n$ for $n \geq 0$ and $0 \leq k \leq n$.

We denote by j_n the inclusion map $\Delta^0 \cong \{0\} \subset \Delta^n$ for $n \geq 0$.

We denote by t_n the unique map from Δ^n to Δ^0 for $n \geq 0$.

We denote by $(\Delta^n)'$ the nerve of the groupoid with $n+1$ canonically isomorphic objects.

We denote by u_n the inclusion map $\Delta^0 \subset (\Delta^n)'$ for $n \geq 0$. There is a unique such map up to isomorphism.

We denote by I_n the union of the segments corresponding to $\{k, k+1\}$ for $0 \leq k \leq n-1$ in Δ^n for $n \geq 2$.

We denote by i_n the inclusion map $I_n \subset \Delta^n$ for $n \geq 0$. We call them the spine maps.

2.2 The Quillen model structure

In this section we present the Quillen model structure on simplicial sets, which represents weak homotopy types. By that we mean that there is a Quillen equivalence between the Quillen model structure and a model structure on topological spaces with weak homotopy equivalences as weak equivalences. This result is presented for example in the first chapter of Goerss and Jardine's book [5].

Now we describe this model structure.

Definition 2.1. *A Kan fibration is a map which has the right lifting property against the horns h^n_k for $n > 0$ and $0 \leq k \leq n$.*

A Kan complex is a simplicial set K such that the unique map from K to Δ^0 is a Kan fibration.

An anodyne extension is a map which has the left lifting property against all Kan fibrations.

A trivial fibration is a map which has the right lifting property against the δ^n for $n \geq 0$.

Lemma 2.2. *The monomorphisms form the smallest saturated class of morphisms containing the δ^n for $n \geq 0$.*

Corollary 2.3. *A trivial fibration has the right lifting property against any monomorphism.*

Proof. By Lemma A.5, the class of morphisms having the left lifting property against the trivial fibrations is saturated. But it contains the δ^n for $n \geq 0$ by definition, therefore it contains all the monomorphisms by Lemma 2.2. □

Lemma 2.4. *The anodyne extensions form the smallest saturated class of morphisms containing the horns h^n_k for $n > 0$ and $0 \leq k \leq n$.*

Proof. This is a consequence of Lemma A.8. □

The category of simplicial sets is a presheaf category, and therefore it is cartesian closed. We denote by $\underline{\mathrm{Hom}}_{\mathbf{S}}(X, Y)$ the mapping simplicial set for X and Y simplicial sets. This mapping simplicial set will also be denoted by Y^X.

We denote by $\pi_0 : \mathbf{S} \to \mathrm{Set}$ the left adjoint to the inclusion of sets in simplicial sets.

Definition 2.5. *A map of simplicial set $f : X \to Y$ is called a weak homotopy equivalence if for all Kan complex K the induced map :*

$$\pi_0(f^*) : \pi_0(\underline{\mathrm{Hom}}_{\mathbf{S}}(Y, K)) \to \pi_0(\underline{\mathrm{Hom}}_{\mathbf{S}}(X, K))$$

is a bijection.

A map of simplicial sets is a weak homotopy equivalence if and only if its geometric realisation is a weak homotopy equivalence in the usual sense.

Theorem 2.6. *There exists a model structure on $\mathbf{S}$ called the Quillen model structure where :*

- *The weak equivalences are the weak homotopy equivalences.*
- *The cofibrations are the monomorphisms.*
- *The acyclic cofibrations are the anodyne extensions.*
- *The fibrations are the Kan fibrations.*
- *The acyclic fibrations are the trivial fibrations.*

One of the key property which needs to be proven in order to establish this theorem is that the Kan fibrations which are weak homotopy equivalences are precisely the trivial fibrations.

Note that the fibrant objects in the Quillen model structure are precisely the Kan complexes. Therefore it should be possible to interpret a Kan complex X as an ∞-groupoid $\mathcal{C}$. In fact we can interpret X_0 as the set of objects in $\mathcal{C}$, the set X_1 as the set of morphisms in $\mathcal{C}$, and X_n for $n \geq 2$ as the set commutative n-simplices in $\mathcal{C}$. Then the right lifting properties against h^n_k for $n > 0$ and $0 \leq k \leq n$ encode the relevant structure for $\mathcal{C}$ to be an ∞-groupoid.

Definition 2.7. *Let $u : A \to B$, $v : A' \to B'$ and $f : X \to Y$ be maps of simplicial sets.*

Then we denote by $< u, f >$ the induced map :

$$< u, f >: X^B \to Y^B \times_{Y^A} X^A$$

and we denote by $u \times' v$ the induced map :

$$u \times' v : B \times A' \coprod_{A \times A'} A \times B' \to B \times B'$$

Lemma 2.8. *Let $u : A \to B$ and $v : A' \to B'$ be monomorphisms and let $f : X \to Y$ be a Kan fibration. Then :*

- *$< u, f >$ is a Kan fibration which is a weak homotopy equivalence whenever u or f is.*
- *$u \times' v$ is a monomorphism which a weak homotopy equivalence whenever u or v is.*

In the language of Appendix D.3, this means that the cartesian product of simplicial set is a left Quillen bifunctor for the Quillen model structure.

2.3 The Joyal model structure

Now we present the Joyal model structure.

Definition 2.9. *A mid-fibration is a map which has the right lifting property against the horns h^n_k for $n > 1$ and $0 < k < n$.*

A quasi-category is a simplicial set S such that the unique map from S to Δ^0 is a mid-fibration.

A mid-anodyne extension is a map which has the left lifting property against mid-fibrations.

Lemma 2.10. *The mid-anodyne extensions form the smallest saturated class of morphisms containing the horns h^n_k for $n > 1$ and $0 < k < n$.*

Proof. This is a consequence of Lemma A.8. □

As expected the usual categories can be seen as $(\infty, 1)$-categories, i.e. as quasi-categories. We use Appendix A.3 on adjunctions from a presheaf category.

Definition 2.11. *We define the functor $\tau_1 : \mathbf{S} \to \mathrm{Cat}$ preserving colimits by defining $\tau_1(\Delta^n)$ to be the ordinal n seen as a category.*

Its right adjoint is denoted by $N : \mathrm{Cat} \to \mathbf{S}$ and is called the nerve functor.

Lemma 2.12. *The nerve functor is full and faithful.*

We denote by $\tau_0(X)$ with X a simplicial set the isomomorphism classes of objects in $\tau_1(X)$. This τ_0 is a functor from simplicial sets to sets.

Definition 2.13. *A map $f : X \to Y$ of simplicial sets is called a weak categorical equivalence if for all quasi-category S the induced map :*

$$\tau_0(f^*) : \tau_0(\underline{\mathrm{Hom}}_{\mathbf{S}}(Y, S)) \to \tau_0(\underline{\mathrm{Hom}}_{\mathbf{S}}(X, S))$$

is a bijection.

Theorem 2.14. *There exists a model structure on $\mathbf{S}$ called the Joyal model structure where :*

- *The cofibrations are the monomorphims.*
- *The weak equivalences are the weak categorical equivalences.*
- *The fibrant objects are the quasi-categories.*

The fibrations in this model structure are called the categorical fibrations, and the monomorphisms which are weak categorical equivalences are called the acyclic categorical cofibrations.

The interpretation of a quasi-category as an $(\infty, 1)$-category is similar to the interpretation of a Kan complex as an ∞-groupoid. The fact that we do not ask for every morphism between objects to be weakly invertible in an $(\infty, 1)$-category is encoded by quasi-categories having the left lifting property against fewer horns than Kan complexes. In fact for X a simplicial set we can define an element of X_1 to be weakly invertible if its image in $\tau_1(X)$ is an isomorphism. Then it can be proven that a quasi-category X is a Kan complex if and only if all elements of X_1 are weakly invertible. This formalises the principle that ∞-groupoids are $(\infty, 1)$-categories with weakly invertible morphisms between objects.

Lemma 2.15. *The mid-anodyne extensions are acyclic categorical cofibrations. The categorical fibrations are mid-fibrations.*

Note that the converses fail, even for maps between quasi-categories. But we have a partial result in the other direction. We denote by $u_1 : \Delta^0 \subset (\Delta^1)'$ the inclusion of an object in $(\Delta^1)'$. We do not specify the object because both choices lead to isomorphic maps.

Lemma 2.16. *A map between quasi-categories is a categorical fibration if and only if it is a mid-fibration which have the right lifting property against $u_1 : \Delta^0 \subset (\Delta^1)'$.*

Lemma 2.17. *Let $u : A \to B$ and $v : A' \to B'$ be monomorphisms and let $f : X \to Y$ be a categorical fibration. Then :*

- *$< u, f >$ is a categorical fibration which is a weak categorical equivalence whenever u or f is.*
- *$u \times' v$ is a monomorphism which is a weak categorical equivalence whenever u or v is.*

In the language of Appendix D.3, this means that the cartesian product of simplicial sets is a left Quillen bifunctor for the Joyal model structure.

2.4 The largest Kan sub-complex in a quasi-category

It is clear that Kan complexes are quasi-categories.

Lemma 2.18. *There exists a right adjoint to the inclusion of Kan complexes in quasi-categories. It is denoted by J.*

In order to understand this adjoint, recall that Kan complexes are precisely the quasi-categories with all morphisms weakly invertible. Henceforth $J(X)$ is the largest Kan sub-complex of X, meaning that it is obtained from X by discarding the non-weakly invertible morphisms.

Lemma 2.19. *The functor J takes weak categorical equivalences to weak homotopy equivalences.*

The functor J takes categorical fibrations to Kan fibrations.

Lemma 2.20. *Let X be a quasi-category. The functor $A \mapsto J(X^A) : \mathbf{S} \to \mathbf{S}^{op}$ has a right adjoint denoted by $A \mapsto X^{(A)} : \mathbf{S}^{op} \to \mathbf{S}$.*

Lemma 2.21. *The functor $A \mapsto X^{(A)}$ takes weak homotopy equivalences to weak categorical equivalences.*

2.5 Two Quillen adjunctions between the Quillen and Joyal model structures

Lemma 2.22. *Weak categorical equivalences are weak homotopy equivalences, so that the identity adjunction is a Quillen adjunction from the Joyal model structure to the Quillen model structure.*

In this situation we say that the Quillen model structure is a Bousfield localisation of the Joyal model structure, see Appendix D.6 for a definition.

We denote by $(\Delta^n)'$ the nerve of the groupoid with $n+1$ canonically isomorphic objects. In order to build the next adjunction, we use the results of Appendix A.3, on the adjunctions with domain a presheaf category.

Definition 2.23. *We define a functor $k_! : \mathbf{S} \to \mathbf{S}$ preserving colimits by $k_!(\Delta^n) = (\Delta^n)'$. It has a right adjoint denoted by $k^!$.*

Note that intuitively $k^!$ is not far from an extension of J to all simplicial sets. This can be formalised : we have natural trivial fibrations $k^!(X) \to J(X)$ for any quasi-category X.

Lemma 2.24. *The adjunction $k_! : \mathbf{S} \to \mathbf{S} : k^!$ is a Quillen adjunction from the Quillen model structure to the Joyal model structure.*

2.6 Some lemmas about simplicial sets

We list various results about simplicial sets.

Lemma 2.25. *The maps $j_n : \Delta^0 \cong \{0\} \subset \Delta^n$ are anodyne for $n \geq 0$*

Lemma 2.26. *The spine maps i_n are mid-anodyne for $n \geq 2$.*

Lemma 2.27. *The spine maps i_n for $n \geq 2$ and the map $u_1 : \Delta^0 \subset (\Delta^1)'$ are acyclic categorical cofibrations.*

Lemma 2.28. *Let A be some saturated class of monomorphisms in $\mathbf{S}$. Assume it has the right cancellation property (i.e. $uv \in A$ and $v \in A$ implies $u \in A$). If i_n is in A for $n \geq 2$, then A contains every mid-anodyne extension.*

3 Bisimplicial sets

We denote by $\mathbf{S}^{(2)}$ the category of bisimplicial sets. In this section we present some features of this category.

3.1 Relations with simplicial sets

For X a bisimplicial set we denote by $X_{m,\bullet}$ the simplicial set $k \mapsto X_{m,k}$. It is called the m-th row of X. Similarly we denote by $X_{\bullet,n}$ the simplicial set $k \mapsto X_{k,n}$. It is called the n-th column of X.

Definition 3.1. *We define* $\_\Box\_ : \mathbf{S}\times\mathbf{S}\to\mathbf{S}^{(2)}$ *by* $(A\Box B)_{m,n} = A_m \times B_n$ *with the obvious face and degeneracy maps.*

We define $\_\backslash\_ : \mathbf{S}^{op}\times\mathbf{S}^{(2)}\to\mathbf{S}$ *by* $(A\backslash X)_n = \mathrm{Hom}_{\mathbf{S}}(A, X_{\bullet,n})$.

We define $\_/\_ : \mathbf{S}^{(2)}\times\mathbf{S}^{op}\to\mathbf{S}$ *by* $(X/B)_n = \mathrm{Hom}_{\mathbf{S}}(B, X_{n,\bullet})$.

We note that the representable bisimplicial sets are the $\Delta^m\Box\Delta^n$ for $m,n\geq 0$. Moreover $\Delta^n\backslash X = X_{\bullet,n}$ is the n-th column of X and $X/\Delta^n = X_{n,\bullet}$ is its n-th row.

Lemma 3.2. *For all A and B simplicial sets and X a bisimplicial set, we have the following natural isomorphisms :*

$$\mathrm{Hom}_{\mathbf{S}}(A, X/B)\cong \mathrm{Hom}_{\mathbf{S}^{(2)}}(A\Box B, X)\cong \mathrm{Hom}_{\mathbf{S}}(B, A\backslash X)$$

Proof. We use basic properties of ends, as presented in [9]. We have the following string of natural isomorphisms :

$$\mathrm{Hom}_{\mathbf{S}}(A, X/B)\cong \int_k \mathrm{Hom}_{\mathrm{Set}}(A_k, \mathrm{Hom}_{\mathbf{S}}(B, X_{k,\bullet}))$$

$$\cong \int_k \mathrm{Hom}_{\mathrm{Set}}\left(A_k, \int_l \mathrm{Hom}_{\mathrm{Set}}(B_l, X_{k,l})\right)\cong \int_{k,l}\mathrm{Hom}_{\mathrm{Set}}(A_k, \mathrm{Hom}_{\mathrm{Set}}(B_l, X_{k,l}))$$

$$\cong \int_{k,l}\mathrm{Hom}_{\mathrm{Set}}(A_k\times B_l, X_{k,l})\cong \mathrm{Hom}_{\mathbf{S}^{(2)}}(A\Box B, X)$$

The other natural isomorphism is proved similarly. □

The situation of Lemma 3.2 is described abstractly in Appendix A.2, where the notations $<\_\backslash\_>$, $<\_/\_>$ and $\_\Box'\_$ are introduced.

Definition 3.3. *We denote by $p_1, p_2 : \Delta^2\to\Delta$ the projections. We denote by $i_1, i_2 : \Delta\to\Delta^2$ the inclusions given by $i_1(n) = (n,0)$ and $i_2(n) = (0,n)$. We have adjunctions :*

$$p_1^* : \mathbf{S}\to\mathbf{S}^{(2)} : i_1^*$$

$$p_2^* : \mathbf{S}\to\mathbf{S}^{(2)} : i_2^*$$

Proof. For K a simplicial set and X a bisimplicial set, we have the natural isomorphisms :

$$p_1^*(K)\cong K\Box\Delta^0$$

$$i_1^*(X)\cong X/\Delta^0$$

$$p_2^*(K)\cong \Delta^0\Box K$$

$$i_2^*(X)\cong \Delta^0\backslash X$$

Then we can use Lemma 3.2. □

Lemma 3.4. *Let X be a simplicial set and let Y be a bisimplicial set. Let $u : X \to i_1^*(Y)$ be a morphism and let $v : p_1^*(X) \to Y$ be the morphism obtained from u by using the adjunction.*

Then for all $n \geq 0$, the morphism v/Δ^n is equal to the composite :

$$p_1^*(X)/\Delta^n \cong X \xrightarrow{u} i_1^*(Y) \cong Y/\Delta^0 \xrightarrow{Y/t_n} Y/\Delta^n$$

with t_n the unique morphism from Δ^n to Δ^0.

3.2 Simplicial enrichment

See Appendix D.4 for the definitions of simplicial enrichments and their tensors and cotensors.

Definition 3.5. *We define :*

$$\underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(X, Y) = i_2^*(Y^X)$$

This defines an enrichment of $\mathbf{S}^{(2)}$ in simplicial sets.

Lemma 3.6. *This simplicial enrichment admits tensors and cotensors.*

Proof. Let X and Y be bisimplicial sets and S be a simplicial set, we have the natural isomorphims :

$$\mathrm{Hom}_{\mathbf{S}}(S, \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(X, Y)) = \mathrm{Hom}_{\mathbf{S}}(S, i_2^*(Y^X)) \cong \mathrm{Hom}_{\mathbf{S}^{(2)}}(p_2^*(S), Y^X)$$

$$\cong \mathrm{Hom}_{\mathbf{S}^{(2)}}(X \times p_2^*(S), Y) \cong \mathrm{Hom}_{\mathbf{S}^{(2)}}(X, Y^{p_2^*(S)})$$

$\square$

3.3 Trivial fibrations

Recall that for $m \geq 0$, we denote by δ^m the inclusion $\partial\Delta^m \subset \Delta^m$.

Definition 3.7. *A map of bisimplicial sets is a called a trivial fibration if it has the right lifting property against $\delta^m \Box' \delta^n$ for all $m, n \geq 0$.*

Lemma 3.8. *The class of monomorphisms of bisimplicial sets is the smallest saturated class of morphisms containing $\delta^m \Box' \delta^n$ for all $m, n \geq 0$.*

Corollary 3.9. *A trivial fibration has the right lifting property against any monomorphism.*

Proof. By Lemma A.5, the class of morphisms having the left lifting property against the trivial fibrations is saturated. But it contains the $\delta^m \Box' \delta^n$ for $m, n \geq 0$ by definition, therefore it contains all the monomorphisms by Lemma 3.8. $\square$

4 Model structures on bisimplicial sets

Now we present four model structures on bisimplicial sets. The vertical and horizontal model structures are build using the theory of Reedy model structures. This theory is presented in Appendix C in the special case of bisimplicial sets. The Segal and Rezk model structures are Bousfield localisations of the vertical model structure, and are build using results from Appendix D.7.

4.1 The vertical model structure

Theorem 4.1. *There exists a model structure on bisimplicial sets called the vertical model structure where :*

- *The cofibrations are the monomorphisms.*
- *The weak equivalences are the column-wise weak homotopy equivalence, i.e. the maps f such that for all $n \geq 0$ the map $\Delta^n \backslash f$ is a weak homotopy equivalence.*
- *The fibrations are the maps f such that for all $m \geq 0$, the map $< \delta^m \backslash f >$ is a Kan fibration.*
- *The acyclic fibrations are the maps f such that for all $m \geq 0$, the map $< \delta^m \backslash f >$ is a trivial fibration.*

Proof. We use Lemma C.1 and the Quillen model structure. All that is left to prove is that the cofibrations are precisely the monomorphisms.

It is enough to show that the acyclic vertical fibrations are the trivial fibrations. A map of bisimplicial set f is a trivial fibration if and only if for all $m, n \geq 0$ we have $\delta^m \square' \delta^n \pitchfork f$, which is equivalent to $< \delta^m \backslash f >$ being a trivial fibration of simplicial sets for all $m \geq 0$ by Lemma A.11. □

See Appendix D.1, D.2 and D.4 for the definitions left proper, combinatorial and simplicial model categories.

Lemma 4.2. *The vertical model structure is left proper, combinatorial and simplicial.*

Proof. See Appendix D.5. □

We will use this lemma in order to build Bousfield localisations of the vertical model structure.

4.2 The horizontal model structure

Theorem 4.3. *There exists a model structure on bisimplicial sets called the horizontal model structure where :*

- *The cofibrations are the monomorphisms.*

- *The weak equivalences are the row-wise weak categorical equivalences, i.e. the maps f such that for all $n \geq 0$ the map f/Δ^n is a weak categorical equivalence.*

- *The fibrations are the maps f such that for all $m \geq 0$, the map $< f/\delta^m >$ is a categorical fibration.*

- *The acyclic fibrations are the maps f such that for all $m \geq 0$, the map $< f/\delta^m >$ is a trivial fibration.*

Proof. We use Lemma C.2. All that is left to prove is that the cofibrations are precisely the monomorphisms.
This can be proved as Theorem 4.1. □

Lemma 4.4. *The adjunction $p_1^* : \mathbf{S} \to \mathbf{S}^{(2)} : i_1^*$ is a Quillen adjunction between the Joyal model structure and the horizontal model structure.*

Proof. It is enough to check that p_1^* is preserves cofibrations and weak equivalences. It clearly preserves monomorphisms, i.e. cofibrations. Moreover we have that $p_1^*(X)/\Delta^n \cong X$ naturally in X for all $n \geq 0$. So if f is a weak categorical equivalence then $p_1^*(f)$ is a row-wise weak categorical equivalence. □

4.3 Bousfield localisations of the vertical model structure

We will now apply the general results summarized in Appendix D.7 in order to build Bousfield localisations of the vertical model structure. The definition of Bousfield localisations can be found in Appendix D.6.

Theorem 4.5. *Assume given a set S of monomorphisms of simplicial sets. Then we can define a model structure on $\mathbf{S}^{(2)}$ where :*

- *The cofibrations are the monomorphisms.*

- *The fibrant objects in the new model structure are the vertically fibrant objects X such that $s\backslash X$ is a weak homotopy equivalence for all s in S.*

- *The weak equivalences are the maps $f : X \to Y$ such that for all fibrant objects Z the induced map :*

$$f^* : \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(Y, Z) \to \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(X, Z)$$

  is a weak homotopy equivalence.

Moreover this model structure is a Bousfield localisation of the vertical model structure.

Proof. By Lemma 4.2, we can apply Theorem D.25 to the set of monomorphisms of bisimplicial set $\{p_1^*(s) \mid s \in S\}$. We obtain a simplicial Bousfield localisation of the vertical model structure where :

- The cofibrations are the monomorphisms.

- The fibrant objects are the vertically fibrant bisimplicial sets X such that for all $s : A \to B$ in S the induced map :

$$(p_1^*(s))^* : \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(p_1^*(B), X) \to \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(p_1^*(A), X)$$

  is a weak homotopy equivalence.

We show that $\underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(p_1^*(A), X)$ is naturally isomorphic to $A\backslash X$, so that the fibrant objects in this model structure are as claimed. But we have the following string of natural isomorphisms for B a simplicial set :

$$\mathrm{Hom}_{\mathbf{S}}(B, \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(p_1^*(A), X)) = \mathrm{Hom}_{\mathbf{S}}(B, i_2^*(X^{p_1^*(A)}))$$

$$\cong \mathrm{Hom}_{\mathbf{S}^{(2)}}(p_2^*(B), X^{p_1^*(A)}) \cong \mathrm{Hom}_{\mathbf{S}^{(2)}}(p_1^*(A) \times p_2^*(B), X)$$
$$\cong \mathrm{Hom}_{\mathbf{S}^{(2)}}(A \Box B, X) \cong \mathrm{Hom}_{\mathbf{S}}(B, A\backslash X)$$

so we can use Yoneda lemma.

All that is left to show in order to conclude is that the weak equivalences are as claimed. But using the fact that every object in this simplicial model structure is cofibrant and Lemma D.13, we obtain the desired characterisation of weak equivalences. $\Box$

Note that the model structures build using this theorem are simplicial.

4.4 The Segal model structure

Recall that for $n \geq 2$ we denote by $i_n : I_n \subset \Delta^n$ the inclusion of the union the edges labelled by $\{k, k+1\}$ for $0 \leq k < n$ in Δ^n.

Definition 4.6. *A Segal space is a vertically fibrant bisimplicial set X such that $i_n\backslash X$ is a weak homotopy equivalence for all $n \geq 2$.*

Theorem 4.7. *There exists a Bousfield localisation of the vertical model structure on bisimplicial sets called the Segal model structure where :*

- *The cofibrations are the monomorphisms.*
- *The fibrant objects are the Segal spaces.*
- *The weak equivalences are the maps $f : X \to Y$ such that for all Segal spaces Z the induced map :*

$$f^* : \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(Y, Z) \to \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(X, Z)$$

  is a weak homotopy equivalence.

Proof. We apply Theorem 4.5 to $\{i_n \mid n \geq 2\}$. $\Box$

Intuitively, for X a bisimplicial set, the simplicial set $X_{\bullet,0}$ will be the space of objects of the $(\infty, 1)$-category it represents, and $X_{\bullet,1}$ will be its space of morphisms. For $n \geq 2$, the fact that $i_n \backslash X$ is a weak homotopy equivalence gives us a canonical weak homotopy equivalence between $X_{\bullet,n}$ and $X_{\bullet,1} \times_{X_{\bullet,0}} \cdots \times_{X_{\bullet,0}} X_{\bullet,1}$ with n copies of $X_{\bullet,1}$, that is the space of strings of n composable morphisms.

Note that we do not use this model structure, but we include it anyway as it is easy to build using Theorem 4.5.

4.5 The Rezk model structure

Recall that $(\Delta^1)'$ is the nerve of the groupoid with two canonically isomorphic objects. Recall that we denote by u_1 the inclusion $\Delta^0 \subset (\Delta^1)'$.

Definition 4.8. *A complete Segal space is a Segal space X such that $u_1 \backslash X$ is a weak homotopy equivalence.*

Theorem 4.9. *There exists a Bousfield localisation of the vertical model structure on bisimplicial sets called the Rezk model structure where :*

- *The cofibrations are the monomorphisms.*
- *The fibrant objects are the complete Segal spaces.*
- *The weak equivalences are the maps $f : X \to Y$ such that for all complete Segal spaces Z the induced map :*

$$f^* : \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(Y, Z) \to \underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}(X, Z)$$

  is a weak homotopy equivalence.

Proof. We apply Theorem 4.5 to $\{i_n \mid n \geq 2\} \cup \{u_1\}$. □

The condition that $u_1 \backslash X$ is a weak homotopy equivalence is analogous to the univalence axiom from homotopy type theory [13]. Intuitively it states that for X a complete Segal space and any two points x and y in $X_{\bullet,0}$ (corresponding to objects in the represented $(\infty, 1)$-category), there is a weak homotopy equivalence between the space of paths between x and y in $X_{\bullet,0}$ and the space of weakly invertible morphisms between x and y in $X_{\bullet,1}$. A precise statement along these lines is Theorem 6.2 in [12].

Lemma 4.10. *A Rezk fibration is a vertical fibration.*

Proof. This is a consequence of the fact that the Rezk model structure is a Bousfield localisation of the vertical model structure, using Lemma D.20. □

5 First results on the model structures on bisimplicial sets

This section collects results about the various model structures introduced in previous section.

5.1 Characterisations of vertical fibrations

Lemma 5.1. *Let $f : X \to Y$ be a map of bisimplicial sets. Then the following are equivalent :*

(i) *f is a vertical fibration, i.e. the map $< \delta^n \backslash f >$ is a Kan fibration for all $n \geq 0$.*

(ii) *$< u \backslash f >$ is a Kan fibration for every monomorphism u.*

(iii) *$< f / h^n_k >$ is a trivial fibration for every $n > 0$ and $0 \leq k \leq n$.*

(iv) *$< f / v >$ is a trivial fibration for every anodyne extension v.*

(v) *f has the right lifting property against $\delta^m \square' h^n_k$ for all $m \geq 0$, $n > 0$ and $0 \leq k \leq n$.*

Proof. By Lemma A.11, we know that the items (i), (iii) and (v) are equivalent.

We show that (i) is equivalent to (ii). It is enough to show that (i) implies (ii). But the class of morphisms u such that $< u \backslash f >$ is a Kan fibration is the class of morphisms such that $u \pitchfork < f / h^n_k >$ for $n > 0$ and $0 \leq k \leq n$, so it is saturated by Lemma A.5, and we can conclude using Lemma 2.2.

We show the equivalence of (iii) and (iv) similarly, using Lemma 2.4. □

5.2 Characterisations of Segal spaces

Lemma 5.2. *Let X be vertically fibrant bisimplicial set. Then the following are equivalent :*

(i) *X is a Segal space, i.e. $i_n \backslash X$ is a weak homotopy equivalence for all $n \geq 2$.*

(ii) *$h^n_k \backslash X$ is a trivial fibration for all $n > 1$ and $0 < k < n$.*

(iii) *$v \backslash X$ is a trivial fibration for all mid-anodyne extension v.*

(iv) *X / δ^n is a mid-fibration for all $n \geq 0$.*

(v) *X / u is a mid-fibration for all monomorphism u.*

(vi) *The unique map from X to the final bisimplicial set has the right lifting property against $h^m_k \square' \delta^n$ for $m > 1$, $0 < k < m$ and $n \geq 0$.*

Proof. By Lemma A.11, we know that the items (ii), (iv) and (vi) are equivalent.

The class of morphisms of simplicial sets v such that $v\backslash X$ is a trivial fibration is the class of v such that $v \pitchfork X\backslash\delta^n$ for $n \geq 0$, so it is saturated by Lemma A.5. So (ii) implies (iii) using Lemma 2.10. So (ii) is equivalent to (iii).

Similarly (iv) implies (v) using Lemma 2.2, and therefore (iv) is equivalent to (v).

By Lemma 2.26 the maps i_n are mid-anodyne extensions, and therefore (iii) implies (i). Now we show the converse. By Lemma 2.28, it is enough to show that for X a vertically fibrant bisimplicial set the class of monomorphisms v such that $v\backslash X$ is a trivial fibration has the two-out-of-three property and contains the i_n for $n \geq 2$. But by Lemma 5.1 we know that $v\backslash X$ is a Kan fibration for any monomorphism v, and therefore it is a trivial fibration if and only if it is a weak homotopy equivalence. But weak homotopy equivalences have the two-out-of-three property and $i_n\backslash X$ is a weak homotopy equivalence for $n \geq 2$ by hypothesis, so we can conclude. □

5.3 A sufficient condition to be a complete Segal space

Lemma 5.3. *Let X be a vertically fibrant simplicial set. Assume that for any acyclic categorical cofibration u, the map $u\backslash X$ is a weak homotopy equivalence. Then X is a complete Segal space.*

Proof. This is consequence of the fact that i_n for $n \geq 2$ and u_1 are acylic categorical cofibrations by Lemma 2.27. □

6 The first Quillen equivalence between the Joyal and Rezk model structures

In this section we show that the adjunction $p_1^* : \mathbf{S} \to \mathbf{S}^{(2)} : i_1^*$ from Definition 3.3 is a Quillen equivalence between the Joyal and Rezk model structures. The definition of Quillen equivalence can be found in Appendix B.3. As i_1^* associates to a bisimplicial set its first row, we call this adjunction the first-row adjunction.

6.1 The first-row Quillen adjunction

In this section we show that the Rezk model structure is a Bousfield localisation of the horizontal model structure. From this we conclude that the first-row adjunction is a Quillen adjunction.

Lemma 6.1. *Let $f : X \to Y$ be a map of bisimplicial sets, and let $u : A \to B$ be a map of simplicial sets such that $u\backslash X$ and $u\backslash Y$ are trivial fibrations. Then $< u\backslash f >$ is a weak homotopy equivalence.*

Proof. By definition of $< u\backslash f >$ we have the following diagram :

$$\begin{tikzcd}
B\backslash X \arrow[dr, "<u\backslash f>"] \arrow[drr, bend left, "u\backslash X"] \arrow[ddr, bend right, "B\backslash f"'] & & \\
& B\backslash Y \times_{A\backslash Y} A\backslash X \arrow[r, "p_{A\backslash X}"] \arrow[d] & A\backslash X \arrow[d, "A\backslash f"] \\
& B\backslash Y \arrow[r, "u\backslash Y"] & A\backslash Y
\end{tikzcd}$$

By hypothesis we know that $u\backslash X$ and $u\backslash Y$ are trivial fibrations, hence so is $p_{A\backslash X}$ as it is the pullback of a trivial fibration. We can conclude that $< u\backslash f >$ is a weak homotopy equivalence by the two-out-of-three property. $\square$

Lemma 6.2. *Let $f : X \to Y$ be a vertical fibration between Segal spaces and let $v : A \to B$ be a mid-anodyne extension of simplicial sets. Then $< v\backslash f >$ is a trivial fibration.*

Proof. By Lemma 5.1, the map $< v\backslash f >$ is a Kan fibration, so it is enough to show that it is a weak homotopy equivalence. By Lemma 6.1 it is enough to show that $v\backslash X$ and $v\backslash Y$ are trivial fibrations, and we conclude using Lemma 5.2. $\square$

Lemma 6.3. *Let $f : X \to Y$ be a vertical fibration between Segal spaces and let $u : A \to B$ be a monomorphism of simplicial sets. Then $< f/u >$ is a mid-fibration between quasi-categories.*

Proof. By Lemma 6.2, the map $< f/u >$ is a mid-fibration, as $v \pitchfork < f/u >$ with v mid-anodyne is equivalent to $u \pitchfork < v\backslash f >$.

Now Lemma 5.2 shows that the domain of $< f/u >$ (namely X/B) is a quasi-category, as $X/\emptyset$ is the terminal object and $\emptyset \to B$ is a monomorphism.

We have the following pullback square :

$$\begin{tikzcd}
Y/B \times_{Y/A} X/A \arrow[r, "p_{X/A}"] \arrow[d] & X/A \arrow[d, "f/A"] \\
Y/B \arrow[r, "Y/u"'] & Y/A
\end{tikzcd}$$

By Lemma 5.2 we know that X/A is a quasi-category and that Y/u is a mid-fibration. But then $p_{X/A}$ is a mid-fibration as well, and then the codomain of $< f/u >$ is a quasi-category. $\square$

Lemma 6.4. *Let $f : X \to Y$ be a vertical fibration between complete Segal spaces and let $u : A \to B$ be a monomorphism of simplicial sets. Then $< f/u >$ is a categorical fibration.*

Proof. By Lemma 6.3, the map $< f/u >$ is a mid-fibration between quasi-categories. Hence by Lemma 2.16 it is enough to check that $< f/u >$ has the right lifting property against $u_1 : \Delta^0 \subset (\Delta^1)'$. But $u_1 \pitchfork < f/u >$ is equivalent to $u \pitchfork < u_1 \backslash f >$, so it is enough to check that $< u_1 \backslash f >$ is a trivial fibration.

By Lemma 5.1, the map $< u_1 \backslash f >$ is a Kan fibration. So it is enough to show that it is a weak homotopy equivalence. So by Lemma 6.1 we just need to prove that $u_1 \backslash X$ and $u_1 \backslash Y$ are trivial fibrations.

We check that $u_1 \backslash Z$ is a trivial fibration for any complete Segal space Z. We know that Z is vertically fibrant so $u_1 \backslash Z$ is a Kan fibration by Lemma 5.1. Moreover it is a weak homotopy equivalence by the definition of a complete Segal space. □

The next theorem is fundamental, and will be used several times in what follows.

Theorem 6.5. *The Rezk model structure is a Bousfield localisation of the horizontal model structure.*

Proof. By Lemma B.7 it is enough to check that the identity functors form a Quillen adjunction. Both model structures have the monomorphisms as cofibrations by Theorems 4.1 and 4.9. By Lemma B.6 it is enough to check that a Rezk fibration between complete Segal spaces is an horizontal fibration.

But we know that a Rezk fibration is a vertical fibration by Lemma 4.10, so we can conclude using Lemma 6.4. □

Now we can see that the first-row adjunction is a Quillen adjunction.

Lemma 6.6. *The adjunction $p_1^* : \mathbf{S} \to \mathbf{S}^{(2)} : i_1^*$ is a Quillen adjunction between the Joyal model structure and the Rezk model structure.*

Proof. This is a consequence of Lemma 4.4 and Theorem 6.5. □

6.2 The first-row Quillen adjunction is a homotopy localisation

See Appendix B.3 for the definition of homotopy localisation. We introduce some terminology for bisimplicial sets. Recall that for $n \geq 0$, we denote by t_n the unique map from Δ^n to Δ^0.

Definition 6.7. *A bisimplicial set X is said categorically constant if for all $n \geq 0$, the map $X/t_n : X/\Delta^0 \to X/\Delta^n$ is a weak categorical equivalence.*

Lemma 6.8. *A vertically fibrant simplicial set is categorically constant.*

Proof. Let us denote by j_n the map $\Delta^0 \cong \{0\} \subset \Delta^n$. Assuming X vertically fibrant, we need to show that X/t_n is a weak categorical equivalence. By Lemma 5.1, we know that X/j_n is a trivial fibration since j_n is anodyne by Lemma 2.25, so X/j_n is a weak categorical equivalence. Then $(X/j_n)(X/t_n) = X/(t_n j_n) = \mathrm{id}_{X/\Delta^0}$ is a weak categorical equivalence and we can conclude using the two-out-of-three property. $\square$

Lemma 6.9. *The Quillen adjunction $p_1^* : \mathbf{S} \to \mathbf{S}^{(2)} : i_1^*$ between the Joyal model structure and the Rezk model structure is a homotopy localisation.*

Proof. By Lemma B.10 it is enough to show that the counit $\epsilon_X : p_1^* i_1^*(X) \to X$ is a Rezk weak equivalence for any complete Segal space X, since we can choose the identity as cofibrant replacement.

By Theorem 6.5 it is enough to show that ϵ_X is a row-wise categorical equivalence. But ϵ_X/Δ^n is isomorphic to X/t_n by Lemma 3.4, which is a weak categorical equivalence by Lemma 6.8. $\square$

6.3 The first-row Quillen adjunction is a homotopy colocalisation

Here we need to build explicitly fibrant remplacements for bisimplicial sets of the form $p_1^*(X)$ with X a quasi-category.

Definition 6.10. *Let X be a quasi-category, then we define a bisimplicial set $\Gamma(X) : n \mapsto J(X^{\Delta^n})$ using the cosimplicial structure of the $(\Delta^n)_{n\geq 0}$.*

Lemma 6.11. *Let X be a quasi-category, then we have isomorphisms*

$$A\backslash\Gamma(X) \cong J(X^A)$$

and

$$\Gamma(X)/A \cong X^{(A)}$$

natural in A in $\mathbf{S}^{op}$.

Proof. We show the first isomorphism. Both functors $A \mapsto A\backslash\Gamma(X)$ and $A \mapsto J(X^A)$ are naturally isomorphic on representable simplicial sets. So by Lemma A.13 it is enough to show that they are colimit-preserving in order to conclude. But both have right adjoints by Lemmas 2.20 and 3.2.

The second isomorphism comes from the unicity of adjoints. $\square$

Definition 6.12. *Let X be a quasi-category. We know that*

$$i_1^*(\Gamma(X)) = \Gamma(X)/\Delta^0 \cong X^{(\Delta^0)} \cong X$$

so by adjunction we obtain a morphism $p_1^*(X) \to \Gamma(X)$.

Lemma 6.13. *Let X be a quasi-category. The morphism $p_1^*(X) \to \Gamma(X)$ of Definition 6.12 is a Rezk weak equivalence, and $\Gamma(X)$ is a complete Segal space.*

Proof. We need to prove that $\Gamma(X)$ is vertically fibrant, i.e. $\delta^n\backslash\Gamma(X)$ is a Kan fibration for all $n \geq 0$. By Lemma 6.11 these maps are isomorphic to $J(X^{\delta^n})$. But X^{δ^n} is a categorical fibration by Lemma 2.17, so $J(X^{\delta^n})$ is a Kan fibration by Lemma 2.19.

Now we need to prove that $\Gamma(X)$ is a complete Segal space. By Lemma 5.3, it is enough to show that for any acyclic categorical cofibration u, the map $u\backslash\Gamma(X)$ is a weak homotopy equivalence. This map is isomorphic to $J(X^u)$ by Lemma 6.11. But X^u is a weak categorical equivalence by Lemma 2.17. Therefore by Lemma 2.19, the map $J(X^u)$ is a weak homotopy equivalence.

Now we prove that the morphism $p_1^*(X) \to \Gamma(X)$ is a Rezk weak equivalence. By Theorem 6.5 it is enough to show that it is a row-wise weak categorical equivalence. But by Definition 6.12 and Lemma 3.4 it is enough to show that $\Gamma(X)/t_n$ is a weak categorical equivalence for all $n \geq 0$, with t_n the unique map from Δ^n to Δ^0. But this map is isomorphic to $X^{(t_n)}$ by Lemma 6.11 and we can conclude using Lemma 2.21, because t_n is a weak homotopy equivalence. $\square$

Lemma 6.14. *The Quillen adjunction $p_1^* : \mathbf{S} \to \mathbf{S}^{(2)} : i_1^*$ is a homotopy colocalisation.*

Proof. Using Lemma B.11 with the fibrant replacement of Lemma 6.13, it is enough to show that for any quasi-category X the composite map :

$$X \to i_1^* p_1^*(X) \to i_1^*\Gamma(X)$$

is a weak categorical equivalence. By Definition 6.12 this composite is an isomorphism, hence a weak categorical equivalence. $\square$

Now we can combine the results of this section in order to obtain the first-row Quillen equivalence.

Theorem 6.15. *The adjunction $p_1^* : \mathbf{S} \to \mathbf{S}^{(2)} : i_1^*$ is a Quillen equivalence between the Joyal model structure and the Rezk model structure.*

Proof. This is a consequence of Lemmas 6.9, 6.14 and B.12. $\square$

One consequence of this theorem is that a complete Segal space is determined up to weak equivalence by its first row.

7 The second Quillen equivalence between the Joyal and Rezk model structures

Now we show that there is a Quillen equivalence in the other direction, from the Rezk model structure to the Joyal model structure.

7.1 The spaces of simplices adjunction

We will use the results from Appendix A.3 on functors from a presheaf category.

Definition 7.1. *We define the functor $t_!$: $\mathbf{S}^{(2)} \to \mathbf{S}$ preserving colimits by $t_!(\Delta^m \Box \Delta^n) = \Delta^m \times (\Delta^n)'$ for $m, n \geq 0$. It has a right adjoint denoted by $t^!$.*

Note that $t^!(X)_{m,n} = \mathrm{Hom}_{\mathbf{S}}(\Delta^m \times (\Delta^n)', X)$, as claimed in the introduction. Since $t^!(X)_{m,\bullet}$ is a good model for the space of m-simplices in X, we call this adjunction the spaces of simplices adjunction.

Lemma 7.2. *We have the following natural isomorphisms, for A and B simplicial sets and X a bisimplicial set :*

$$t_!(A \Box B) \cong A \times k_!(B)$$

$$A \backslash t^!(X) \cong k^!(X^A)$$

Proof. The first isomorphism is easy to check for A and B representable, and both functors commute with colimits in each variable, so we can use Lemma A.13. The second isomorphism is a consequence of the first one using Yoneda lemma. $\Box$

Lemma 7.3. *Let $u : A \to B$ and $v : A' \to B'$ be maps of simplicial sets, and let $f : X \to Y$ be a map of bisimplicial sets. Then $t_!(u \Box' v) = u \times' k_!(v)$ and $< u \backslash t^!(f) >= k^!(< u, f >)$.*

Proof. This is a consequence of Lemma 7.2. $\Box$

7.2 The spaces of simplices Quillen equivalence

Our strategy is as follows : first we show that the spaces of simplices adjunction is a Quillen adjunction from the vertical model structure to the Joyal model structure. Then we show that it extends to the Rezk model structure. Finally we will use the two-out-of-three property of Quillen equivalences and the results of the previous section on the first-row adjunction in order to show that the spaces of simplices adjunction is a Quillen equivalence.

Lemma 7.4. *The adjunction $t_! : \mathbf{S}^{(2)} \to \mathbf{S} : t^!$ is a Quillen adjunction between the vertical model structure and the Joyal model structure.*

Proof. By Lemma A.6 the class of morphisms of bisimplicial sets u such that $t_!(u)$ is a monomorphism is saturated, because $t_!$ preserves colimits. By Lemma 3.8 it is enough to show that $t_!(\delta^m \Box' \delta^n)$ is a monomorphism for $m, n \geq 0$ in order to conclude that $t_!$ preserves cofibrations. But by Lemma 7.3, we know that $t_!(\delta^m \Box' \delta^n) = \delta^m \times' k_!(\delta^n)$, and we can conclude using Lemmas 2.17 and 2.24.

Now we show that $t^!$ preserves fibrations. Let $f : X \to Y$ be a categorical fibration, we need to show that $t^!(f)$ is a vertical fibration, i.e. that for all monomorphism u of simplicial set the map $< u \backslash t^!(f) >$ is a Kan fibration. But

$< u\backslash t^!(f) >= k^!(< u, f >)$ by Lemma 7.3, $< u, f >$ is a categorical fibration by Lemma 2.17 and $k^!$ sends categorical fibrations to Kan fibrations by Lemma 2.24. □

Lemma 7.5. *The adjunction $t_! : \mathbf{S}^{(2)} \to \mathbf{S} : t^!$ is a Quillen adjunction between the Rezk model structure and the Joyal model structure.*

Proof. By Lemmas 7.4 and D.22, it is enough to show that $t^!$ sends quasi-category to complete Segal spaces. Let X be a quasi-category. We know that $t^!(X)$ is vertically fibrant by Lemma 7.4, so by Lemma 5.3 it is enough to prove that $u\backslash t^!(X)$ is a weak homotopy equivalence for any acyclic categorical cofibration u. By Lemma 7.2, we have that $u\backslash t^!(X) = k^!(X^u)$. We know that X^u is a tivial fibration by Lemma 2.17, and that $k^!$ preserves trivial fibrations by Lemma 2.24. □

Theorem 7.6. *The adjunction $t_! : \mathbf{S}^{(2)} \to \mathbf{S} : t^!$ is a Quillen equivalence between the Rezk model structure and the Joyal model structure.*

Proof. By Lemma 7.5 we know that it is a Quillen adjunction. Using Lemma B.13 and Theorem 6.15, it is enough to prove that the composed adjunction $t_! p_1^* : \mathbf{S} \to \mathbf{S} : i_1^* t^!$ is a Quillen equivalence.

We have the natural isomorphisms :

$$t_! p_1^*(A) \cong t_!(A \Box \Delta^0) \cong A \times k_!(\Delta^0) \cong A$$

for A a simplicial set. But any adjunction naturally isomorphic to the identity adjunction is a Quillen equivalence. □

8 Conclusion

The vertical model structure is defined from the Quillen model structure, and then it is suitably localised in order to obtain a model for $(\infty, 1)$-categories. This construction of the Rezk model structure from the Quillen model structure can be seen as a special case of a more general construction of internal $(\infty, 1)$-categories in a given (sufficiently nice) $(\infty, 1)$-category, with the Quillen model structure representing the $(\infty, 1)$-category of ∞-groupoids.

The Rezk model structure is simplicial (as it is obtained using Theorem D.25), but the Joyal model structure is not, and this is not mere coincidence. In fact we have seen that the Rezk model structure is a Bousfield localisation of the horizontal model structure, which is itself build from the Joyal model structure. From this point of view, we can see the construction of the Rezk model structure from the Joyal model structure as a special case of a process replacing a (sufficiently nice) model structure by a Quillen equivalent simplicial model structure, as explained in [3].

Acknowledgement

I heartfully thank Georges Maltsiniotis for giving me many interesting insights about $(\infty, 1)$-categories.

References

[1] Clark Barwick and Daniel M Kan. Relative categories: another model for the homotopy theory of homotopy theories. *Indagationes Mathematicae*, 23(1-2):42–68, 2012.

[2] Julia Bergner. A model category structure on the category of simplicial categories. *Transactions of the American Mathematical Society*, 359(5):2043–2058, 2007.

[3] Daniel Dugger. Replacing model categories with simplicial ones. *Transactions of the American Mathematical society*, pages 5003–5027, 2001.

[4] William G Dwyer and Jan Spalinski. Homotopy theories and model categories. *Handbook of algebraic topology*, 73:126, 1995.

[5] Paul G Goerss and John F Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[6] Mark Hovey. *Model categories*. Number 63. American Mathematical Soc., 2007.

[7] André Joyal. The theory of quasi-categories and its applications. 2008.

[8] André Joyal and Myles Tierney. Quasi-categories vs segal spaces. *arXiv preprint math/0607820*, 2006.

[9] Fosco Loregian. This is the (co) end, my only (co) friend. *arXiv preprint arXiv:1501.02503*, 2015.

[10] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009.

[11] Daniel G Quillen. Homotopical algebra. 1967.

[12] Charles Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3):973–1007, 2001.

[13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

A Definitions and results from category theory

A.1 Left/right lifting properties and saturated classes

Definition A.1. *Let $\mathcal{C}$ be a category. We say that a morphism $i : A \to B$ in $\mathcal{C}$ has the left lifting property against a morphism $f : X \to Y$ in $\mathcal{C}$ if for all commutative squares*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

there exists a dotted arrow h making the two triangles commute.

We say equivalently that f has the right lifting property against i.

We denote this property by $i \pitchfork f$.

Lemma A.2. *Assume given an adjunction $F : \mathcal{C} \to \mathcal{D} : G$. Assume given an arrow u in $\mathcal{C}$ and an arrow v in $\mathcal{D}$. Then we have that $F(u) \pitchfork v$ if and only if $u \pitchfork G(v)$.*

Definition A.3. *Let $\mathcal{C}$ be a category, and let S a class of morphisms in $\mathcal{C}$. We say that S is saturated if :*

- *S is stable by pushouts.*
- *S is stable by retracts.*
- *S is stable by transfinite compositions.*

Definition A.4. *Let H be a class of morphisms in a category $\mathcal{C}$. Then we denote by $l(H)$ (resp. $r(H)$) the class of morphisms which have the left (resp. right) lifting property against all morphisms in H.*

Lemma A.5. *Let H be a class of morphisms in a category $\mathcal{C}$. Then $l(H)$ is saturated.*

Lemma A.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories admitting small colimits. Let $F : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor. If W is a saturated class of morphisms in $\mathcal{D}$, then so is $F^{-1}(W)$.*

Definition A.7. *A pair of classes of morphisms (A, B) in a category $\mathcal{C}$ is called a functorial weak factorisation system if :*

- *There exists a functorial factorisation of any map in $\mathcal{C}$ as a map in A followed by a map in B.*
- *We have $A = l(B)$ and $B = r(A)$.*

Lemma A.8. *Assume given a set of maps S in a presheaf category $\mathcal{C}$. Then $(l(r(S)), r(S))$ is a functorial weak factorisation system. Moreover $l(r(S))$ is the smallest saturated class of morphisms containing S.*

A proof of this last lemma is presented in Appendix A.1.2 of Lurie's book [10]. He calls our saturated classes of morphisms *weakly saturated*. The reasoning used in this proof is called the small object argument.

A.2 Bifunctors divisible on both sides

In this section $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are categories, and $\_\square\_ : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ is a bifunctor.

Definition A.9. *We say that $\_\square\_$ is divisible on both sides if there exists bifunctors $\_\backslash\_ : \mathcal{C}_1^{op} \times \mathcal{C}_3 \to \mathcal{C}_2$ and $\_/\_ : \mathcal{C}_3 \times \mathcal{C}_2^{op} \to \mathcal{C}_1$ such that we have isomorphisms natural in $A \in \mathcal{C}_1^{op}$, $B \in \mathcal{C}_2^{op}$ and $C \in \mathcal{C}_3$:*

$$\mathrm{Hom}_{\mathcal{C}_1}(A, C/B) \cong \mathrm{Hom}_{\mathcal{C}_3}(A\square B, C) \cong \mathrm{Hom}_{\mathcal{C}_2}(B, A\backslash C)$$

Definition A.10. *Assume $\_\square\_$ is divisible on both sides. Let $u : A \to B$ be a morphism in $\mathcal{C}_1$, let $v : A' \to B'$ be a morphism in $\mathcal{C}_2$ and let $f : X \to Y$ be a morphism in $\mathcal{C}_3$.*

Then we denote by $u\square' v$ the induced morphism $A\square B' \coprod_{A\square A'} B\square A' \to B\square B'$.

We denote by $< f/v >$ the induced morphism $X/B' \to Y/B' \times_{Y/A'} X/A'$.

We denote by $< u\backslash f >$ the induced morphism $B\backslash X \to B\backslash Y \times_{A\backslash Y} A\backslash X$.

Lemma A.11. *In the situation of the previous definition we have that $u\square' v \pitchfork f$ if and only if $u \pitchfork < f/v >$ if and only if $v \pitchfork < u\backslash f >$.*

A.3 Adjunction from a presheaf category

For $\mathcal{C}$ and $\mathcal{D}$ two categories, we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$ with natural transformations between them.

Lemma A.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let F be a functor form $\mathcal{C}$ to $\mathcal{D}$. Then there exists a unique up to natural isomorphism colimit-preserving extension of F to $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$. Moreover this extension has a right adjoint G defined by $G(Y)(X) = \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ for X in $\mathcal{C}$ and Y in $\mathcal{D}$.*

This lemma is used to define an adjunction from a presheaf category by specifying a functor on representable presheaves.

We denote by $\mathbf{y}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ the Yoneda embedding. We now state a variant of this lemma.

Lemma A.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then pre-composition with $\mathbf{y}_{\mathcal{C}}$ induces an equivalence between colimit-preserving functors from $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ to $\mathcal{D}$ and $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.*

B Model categories and morphisms between them

B.1 Model categories

We present the notion of model category. The original reference is [11]. Other sources include [4] and [6]. Functorial weak factorisation systems are presented in Definition A.7.

Definition B.1. *A model structure on a category $\mathcal{C}$ admitting all small limits and small colimits is the data of three classes of morphisms (C, W, F) called cofibrations, weak equivalences and fibrations such that :*

- *The weak equivalences satisfy the two-out-of-three property, i.e. if f and g are composable morphisms, then if any two among f, g and $g \circ f$ are in W, so is the third.*
- *The pair $(C \cap W, F)$ is a functorial weak factorisation system.*
- *The pair $(C, F \cap W)$ is a functorial weak factorisation system.*

The maps in $C \cap W$ are called the acyclic cofibrations, and the maps in $F \cap W$ are called the acylic fibrations.

Note that we ask that $\mathcal{C}$ has all small limits and colimits (as opposed to finite limits and colimits), and that the factorisations are required to be functorial. These restrictions are not necessary, but all the model structures considered in this document will satisfy them.

Definition B.2. *Let $\mathcal{C}$ be a model category, then the localisation of $\mathcal{C}$ at the weak equivalences is denoted by* $\mathrm{Ho}(\mathcal{C})$*.*

The category $\mathrm{Ho}(\mathcal{C})$ is called the homotopy category of $\mathcal{C}$.

Definition B.3. *Let $\mathcal{C}$ be a model category. Then an object X in $\mathcal{C}$ is said :*

- *Cofibrant if the unique map from the initial object to X is a cofibration.*
- *Fibrant if the unique map from X to the final object is a fibration.*

For $\mathcal{C}$ a model category, it is known that $\mathrm{Ho}(\mathcal{C})$ is equivalent to the subcategory of fibrant and cofibrant objects in $\mathcal{C}$ with morphisms quotiented by a congruence called the homotopy relation. We do not make this relation explicit as we do not use it.

Definition B.4. *Let $\mathcal{C}$ be a model category. A cofibrant replacement for an object X is a cofibrant object X' together with weak equivalence from X' to X. A fibrant replacement for an object Y is a fibrant object Y' together with weak equivalence from Y to Y'.*

The definition of a model category implies that there exist a cofibrant and a fibrant replacement for any object. These canonical (co)fibrant replacements are functorial, by our definition of model categories.

B.2 Quillen functors

The notion of Quillen adjunction is a notion of morphism between model categories.

Definition B.5. *Let $\mathcal{C}$ and and $\mathcal{D}$ be two model categories. Let $F : \mathcal{C} \to \mathcal{D} : G$ be an adjunction. We say that it is a Quillen adjunction if any of the following equivalent conditions is true :*

- *F preserves cofibrations and acyclic cofibrations.*
- *G preserves fibrations and acyclic fibrations.*
- *F preserves cofibrations and G preserves fibrations.*

In this case we say that F is a left Quillen functor, and that G is a right Quillen functor.

Lemma B.6. *Let $\mathcal{C}$ and and $\mathcal{D}$ be two model categories. Let $F : \mathcal{C} \to \mathcal{D} : G$ be an adjunction. It is a Quillen adjunction if and only if F preserves cofibrations and G sends fibrations between fibrant objects to fibrations.*

Next lemma is sometimes called Ken Brown's lemma.

Lemma B.7. *Let $F : \mathcal{C} \to \mathcal{D} : G$ be a Quillen adjunction. Then F preserves weak equivalences between cofibrant objects.*

B.3 Quillen equivalences

Lemma B.8. *A Quillen adjunction $F : \mathcal{C} \to \mathcal{D} : G$ between model categories $\mathcal{C}$ and $\mathcal{D}$ induces an adjunction $\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D}) : \mathbb{R}G$.*

Definition B.9. *A Quillen adjunction $F : \mathcal{C} \to \mathcal{D} : G$ is called :*

- *A Quillen equivalence if it induces an equivalence of category.*
- *A homotopy localisation if the induced functor $\mathbb{R}G$ is full and faithful.*
- *A homotopy colocalisation if the induced functor $\mathbb{L}F$ is full and faithful.*

It is immediate from the definition that two Quillen equivalent model categories have equivalent homotopy categories. Similarly a homotopy (co)localisation induces a (co)reflection of homotopy categories.

Lemma B.10. *Let $F : \mathcal{C} \to \mathcal{D} : G$ be a Quillen adjunction. Assume that for all fibrant and cofibrant objects X in $\mathcal{D}$ there exists a cofibrant replacement for GX denoted by $l_{GX} : X' \to GX$ such that the composite :*

$$FX' \overset{F(l_{GX})}{\to} FGX \overset{\epsilon_X}{\to} X$$

is a weak equivalence. Then the Quillen adjunction $F : \mathcal{C} \to \mathcal{D} : G$ is a homotopy localisation.

Lemma B.11. *Let $F : \mathcal{C} \to \mathcal{D} : G$ be a Quillen adjunction. Assume that for all fibrant and cofibrant objects Y in $\mathcal{C}$ there exists a fibrant replacement for FY denoted by $r_{FY} : FY \to Y'$ such that the composite :*

$$Y \overset{\eta_Y}{\to} GFY \overset{G(r_{FY})}{\to} GY'$$

is a weak equivalence. Then the Quillen adjunction $F : \mathcal{C} \to \mathcal{D} : G$ is a homotopy colocalisation

Lemma B.12. *A Quillen adjunction is a Quillen equivalence if and only if it is a homotopy localisation and a homotopy colocalisation.*

Lemma B.13. *The Quillen adjunctions have the two-out-of-three property.*

C Reedy model structure

Assume given a model category $\mathcal{D}$. The Reedy model structures are model structures on categories of functors from $\mathcal{C}$ to $\mathcal{D}$ which exist for suitable categories $\mathcal{C}$. One such $\mathcal{C}$ is the category of standard simplices Δ. We present the results for Reedy model structures only for bisimplicial sets, building model structures on bisimplicial sets from model structures on simplicial sets. A general presentation of Reedy model structures can be found in Chapter 5 of Hovey's book [6].

Lemma C.1. *Assume given a model structure M on $\mathbf{S}$. Then we can define a model structure on $\mathbf{S}^{(2)}$ where :*

- *The weak equivalences are the column-wise weak equivalence in M, i.e. the maps f such that for any $n \geq 0$ the map $\Delta^n \backslash f$ is a weak equivalence in M.*
- *The fibrations are the maps f such that for all $n \geq 0$ the map $< \delta^n \backslash f >$ is a fibration in M.*
- *The acyclic fibrations are the maps f such that for all $n \geq 0$ the map $< \delta^n \backslash f >$ is an acyclic fibration in M.*

This result can be seen through a mirror in order to give the following lemma.

Lemma C.2. *Assume given a model structure M on $\mathbf{S}$. Then we can define a model structure on $\mathbf{S}^{(2)}$ where :*

- *The weak equivalence are the row-wise weak equivalence in M, i.e. the maps f such that for any $n \geq 0$ the map f/Δ^n is a weak equivalence in M.*
- *The fibrations are the maps f such that for all $n \geq 0$ the map $< f/\delta^n >$ is a fibration in M.*
- *The acyclic fibrations are the maps f such that for all $n \geq 0$ the map $< f/\delta^n >$ is an acyclic fibration in M.*

D Bousfield localisations of left proper, combinatorial and simplicial model structures

The goal of this appendix is to formulate Theorem D.25.

D.1 Left proper model categories

Definition D.1. *A model category is called left proper if the pushout of a weak equivalence along a cofibration is a weak equivalence.*

Lemma D.2. *If all objects in a model category are cofibrant, then it is left proper.*

D.2 Combinatorial model categories

In this section we give a definition of combinatorial model categories. First we define accessible categories.

Definition D.3. *Let κ be a cardinal and let $\mathcal{C}$ be a category that admits κ-filtered colimits.*

- *An object X in $\mathcal{C}$ is said κ-compact if $\mathrm{Hom}_{\mathcal{C}}(X, \_) : \mathcal{C} \to \mathrm{Set}$ preserves κ-filtered colimits.*
- *$\mathcal{C}$ is said κ-accessible if there exists a set of κ-compact objects which generates $\mathcal{C}$ under κ-filtered colimit.*

A category $\mathcal{C}$ is said accessible if there exists a cardinal κ such that $\mathcal{C}$ is κ-accessible.

The cardinal κ is often required to be regular, but κ-accessible is equivalent to $cof(\kappa)$-accessible, so this restriction is not mandatory.

Definition D.4. *A category is called locally presentable if it is accessible and admits small colimits.*

Lemma D.5. *Presheaf categories are locally presentable.*

Definition D.6. *A model category is said cofibrantly generated if both the classes of cofibrations and acyclic cofibrations are the smallest saturated classes containing a set of morphisms.*

The key fact about this definition is that we require generating *sets* of morphisms.

Definition D.7. *A model category is said combinatorial if it is locally presentable and its model structure is cofibrantly generated.*

D.3 Quillen bifunctors

In this section we assume that $\_\square\_ : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ is a bifunctor divisible on both sides with $\mathcal{C}_i$ a model category for all $i \in \{1, 2, 3\}$. The definition of such a bifunctor can be found in Appendix A.2.

Definition D.8. *The bifunctor* $\_\square\_ : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$ *is called a left Quillen bifunctor if for any cofibration* u *in* $\mathcal{C}_1$ *and cofibration* v *in* $\mathcal{C}_2$ *the map* $u\square' v$ *is a cofibration, which is acyclic if* u *or* v *is.*

The bifunctor $\_\backslash\_ : \mathcal{C}_1^{op} \times \mathcal{C}_3 \to \mathcal{C}_2$ *is called a right Quillen bifunctor if for any cofibration* u *in* $\mathcal{C}_1$ *and fibration* f *in* $\mathcal{C}_3$ *the map* $< u\backslash f >$ *is a fibration, which is acyclic if* u *or* f *is.*

The bifunctor $\_/\_ : \mathcal{C}_3 \times \mathcal{C}_2^{op} \to \mathcal{C}_1$ *is called a right Quillen bifunctor if for any fibration* f *in* $\mathcal{C}_3$ *and cofibration* v *in* $\mathcal{C}_2$ *the map* $< f/v >$ *is a fibration, which is acyclic if* f *or* v *is.*

Note that we have defined a Quillen bifunctor as a bifunctor divisible on both sides, whereas they are usually defined as merely preserving colimits in both variables.

Lemma D.9. *We have that* $\_\square\_$ *is a left Quillen functor if and only if* $\_/\_$ *is a right Quillen functor if and only if* $\_\backslash\_$ *is a right Quillen functor.*

D.4 Simplicial model categories

We define simplicial model categories. We use the notion of Quillen bifunctor from Appendix D.3.

Definition D.10. *A simplicial enrichment of a category* $\mathcal{C}$ *is bifunctor* $\underline{\mathrm{Hom}}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{S}$ *such that* $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)_0 \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *naturally for* X *and* Y *in* $\mathcal{C}$*.*

Definition D.11. *A simplicial enrichment of a category* $\mathcal{C}$ *is said to admit tensors if there exists a bifunctor* $\_ \otimes \_ : \mathbf{S} \times \mathcal{C} \to \mathcal{C}$ *and natural isomorphisms :*

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(K \otimes A, B) \cong \underline{\mathrm{Hom}}_{\mathbf{S}}(K, \underline{\mathrm{Hom}}_{\mathcal{C}}(A, B))$$

for K *a simplicial set and* A *and* B *in* $\mathcal{C}$*.*

It is said to admit cotensors of there exists a bifunctor $\_^{\_} : \mathcal{C} \times \mathbf{S}^{op} \to \mathcal{C}$ *and natural isomorphisms :*

$$\underline{\mathrm{Hom}}_{\mathbf{S}}(K, \underline{\mathrm{Hom}}_{\mathcal{C}}(A, B)) \cong \underline{\mathrm{Hom}}_{\mathcal{C}}(A, B^K)$$

for K *a simplicial set and* A *and* B *in* $\mathcal{C}$*.*

Definition D.12. *Let* $\mathcal{C}$ *be a model category with a simplicial enrichment admitting tensors and cotensors. We say that* $\mathcal{C}$ *is a simplicial model category if* $\underline{\mathrm{Hom}}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{S}$ *is a right Quillen bifunctor, with* $\mathbf{S}$ *given the Quillen model structure.*

Lemma D.13. *Let $\mathcal{C}$ be a simplicial model category. Then a map $u : A \to B$ between cofibrant objects is a weak equivalence if and only if for any fibrant object X the induced map :*

$$u^* : \underline{\mathrm{Hom}}_{\mathcal{C}}(B, X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)$$

is a weak homotopy equivalence.

D.5 The vertical model structure is left proper, combinatorial and simplicial

Lemma D.14. *The vertical model structure is left proper.*

Proof. We know that every object is cofibrant, so we can conclude by Lemma D.2. □

Lemma D.15. *The vertical model structure is combinatorial.*

Proof. As a presheaf catgeory, the category of bisimplicial sets is locally presentable.

The cofibrations are the monomorphisms, hence they are the smallest staurated class of morphisms containing the $\delta^m \Box' \delta^n$ for $m, n \geq 0$ by Lemma 3.8.

The acyclic cofibrations are the maps which have the left lifting property against vertical fibrations. The vertical fibrations are the maps which have the right lifting property against $\delta^m \Box' h^n_k$ for all $m \geq 0$, $n > 0$ and $0 \leq k \leq n$ by Lemma 5.1. So by Lemma A.8 the class of acyclic cofibrations is the smallest saturated class of morphisms containing $\delta^m \Box' h^n_k$ for all $m \geq 0$, $n > 0$ and $0 \leq k \leq n$. □

Lemma D.16. *The internal Hom functor in bisimplicial set is a right Quillen bifunctor for the vertical model structure.*

Proof. By Lemma D.9 it is enough to show that the cartesian product is a left Quillen bifunctor.

We know that for u and v monomorphisms, $u \times' v$ is a monomorphism as well, because this is true for sets. Assume that u or v is a column-wise weak homotopy equivalence. Then $\Delta^n \backslash (u \times' v) \cong (\Delta^n \backslash u) \times' (\Delta^n \backslash v)$ and we can conclude using Lemma 2.8. □

Lemma D.17. *The adjunction $p_2^* : \mathbf{S} \to \mathbf{S}^{(2)} : i_2^*$ is a Quillen adjunction between the Kan model structure and the vertical model structure.*

Proof. It is enough to check that p_2^* preserves cofibrations and weak equivalences. It clearly preserves monomorphisms, i.e. cofibrations. Moreover we have that $\Delta^n \backslash p_2^*(X) \cong X$ naturally in X a simplicial set for all $n \geq 0$. So if f is a weak homotopy equivalence then $p_2^*(f)$ is a column-wise weak homotopy equivalence. □

Lemma D.18. *The vertical model structure is simplicial for the simplicial enrichment of $\mathbf{S}^{(2)}$ of Definition 3.5.*

Proof. We know By Lemma 3.6 that the simplicial enrichment of $\mathbf{S}^{(2)}$ admits tensors and cotensors.

We need to show that $\underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}} : (\mathbf{S}^{(2)})^{op} \times \mathbf{S}^{(2)} \to \mathbf{S}$ is a right Quillen bifunctor. But $\underline{\mathrm{Hom}}_{\mathbf{S}^{(2)}}$ is the composite of the internal Hom bifunctor in $\mathbf{S}^{(2)}$ and i_2^*. The composite of a right Quillen functor and a right Quillen bifunctor is a right Quillen bifunctor, so we can conclude using Lemmas D.16 and D.17. $\square$

D.6 Definition of Bousfield localisations

Definition D.19. *Let $\mathcal{C}$ be a category admitting all small limits and small colimits. Let $M = (C, W, F)$ and $M' = (C', W', F')$ be two model structures on $\mathcal{C}$. We say that M' is a Bousfield localisation of M if $C = C'$ and $W \subset W'$.*

If M' is a Bousfield localisation of M, then the identity adjunction is a homotopy localisation from M to M'.

Lemma D.20. *If $M' = (C', W', F')$ is a Bousfield localisation of $M = (C, W, F)$, then $F' \subset F$.*

Proof. $C \cap W \subset C \cap W'$ implies

$$F' = r(C \cap W') \subset r(C \cap W) = F$$

$\square$

We have a partial converse.

Lemma D.21. *Let $M' = (C', W', F')$ be a Bousfield localisation of $M = (C, W, F)$. Let f be a map between fibrant objects in M'. Then f is in F if and only if it is in F'.*

Lemma D.22. *Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories and let $F : \mathcal{C} \to \mathcal{D} : G$ be a Quillen adjunction. Assume $\mathcal{C}'$ is a Bousfield localisation of $\mathcal{C}$. Then if G takes fibrant objects in $\mathcal{D}$ to fibrant objects in $\mathcal{C}'$, the Quillen adjunction extends to $\mathcal{C}'$.*

Proof. We know that $F : \mathcal{C}' \to \mathcal{D}$ preserves cofibrations, because the cofibrations are the same in $\mathcal{C}$ and $\mathcal{C}'$. By Lemma B.6, it is enough to check that G takes fibrations between fibrant objects to fibrations. Let $f : X \to Y$ be such a fibration between fibrant objects in $\mathcal{D}$. We know that $G(f)$ is a fibration in $\mathcal{C}$, and by hypothesis $G(X)$ and $G(Y)$ are fibrant in $\mathcal{C}'$. We conclude using Lemma D.21. $\square$

D.7 Bousfield localisation at set of cofibrations in a simplicial model category

In this section we present a general mean to build Bousfield localisations of sufficiently good model categories. A possible reference is Appendix A.3.7 in [10].

We denote by L (resp. R) the canonical cofibrant (resp. fibrant) replacement functor given by our definition of model categories.

Definition D.23. *Let $\mathcal{C}$ be a simplicial category. We denote by $\mathrm{Map}_{\mathcal{C}}$ the functor from $\mathcal{C}^{op} \times \mathcal{C}$ to $\mathbf{S}$ which associates to X and Y in $\mathcal{C}$ the simplicial set $\underline{\mathrm{Hom}}_{\mathcal{C}}(LX, RY)$.*

Definition D.24. *Let $\mathcal{C}$ be a simplicial model category, and let S be a set of cofibrations in $\mathcal{C}$.*

- *An object X in $\mathcal{C}$ is called S-local if for all $s : A \to B$ is S, the induced morphism*
$$s^* : \mathrm{Map}_{\mathcal{C}}(B, X) \to \mathrm{Map}_{\mathcal{C}}(A, X)$$
is a weak homotopy equivalence.

- *A morphism $f : X \to Y$ is called an S-equivalence if for all S-local object Z the induced morphism*
$$f^* : \mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$$
is a weak homotopy equivalence.

Theorem D.25. *Let $\mathcal{C}$ be a left proper combinatorial simplicial model category, we denote its model structure by M. Let S be a set of cofibrations in M. Then there is a model structure M' on $\mathcal{C}$ where :*

- *The cofibrations are the cofibrations in M.*
- *The weak equivalences are the S-equivalences.*
- *The fibrant objects in M' are the S-local fibrant objects in M.*

Moreover M' is a Bousfield localisation of M, and it is a simplicial model structure.