Expansion Proofs for Arithmetic

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Abstract

We present two extensions of expansion proofs to arithmetic. We define expansion proofs with induction, and then we define infinitary expansion proofs for arithmetic. We show that expansion proofs with induction are equivalent to Peano Arithmetic. We show that infinitary expansion proofs are a definition of arithmetical truth and show a cut-elimination theorem for them. Moreover, we show that finitary expansion proofs can be translated to infinitary ones, thereby proving consistency of Peano arithmetic.

The general context

Herbrand's theorem [12] is a central theorem for classical first-order logic. It induces that a proof in this case can be represented as the instantiations of its first-order quantifiers. The notion of Herband proof [4] is an example of such a representation. Such a formalism provides a nice balance between easily checkable and compact proofs, and can be generalised e.g. to simple type theory as in [15].

However these formalisms do not support cuts, and it is very natural to try to extend them to this case. Three such extensions have been proposed : proof forests [11], Herbrand nets [14] and expansion proofs [1]. These formalisms draw inspiration both from proof nets and game semantics. The first two only work on prenex formulas. All these formalisms enjoy a weakly normalising cut-elimination procedure, but it is not known whether it is strongly normalising for any of them.

The research problem

We will be concerned here with the extension of expansion proofs to arithmetic. Our main task was to provide in this context a proof of consistency for Peano arithmetic using induction up to ε_0 as only assumption beyond PA.

This is related to the computational content of classical arithmetic, which is a very active subject of research, see for example the survey [2]. Here we try to use expansion proofs in this context as a test for their applicability.

In the end we extract some *interactional* content of proofs in classical arithmetic in the setting of expansion proofs. Indeed expansion proofs present game-like features and our extension is designed to reflect those, hence the word interactional. This is closely related to game semantics for arithmetic [6].

Your contribution

We define expansion proofs with induction and prove they are sound and complete with respect to Peano's arithmetic. This is an extension of pure first-order expansion proofs presented in [1]. We give a correct cut-elimination procedure for them and give some partial termination results.

In order to obtain a full cut-elimination, we used infinitary expansion proofs inspired by Schütte's semi-formal system [18]. They have game-theoretic features, and therefore we call them strategies. We prove them equivalent to the standard sequent calculus for infinitary proofs, and show they have a terminating cut-elimination procedure. Then we give a translation from expansion proofs with induction to strategies, thereby proving consistency of Peano arithmetic.

Arguments supporting its validity

Both our finitary and infinitary expansion proofs are very natural extensions of expansions proofs.

Infinitary expansion proofs for arithmetic are interesting because they shed some light on the *interactional* content of proofs in Peano arithmetic, and clarify the game-like features of expansion proofs for arithmetic. We will argue that they are more natural for arithmetic than finitary expansion proofs.

On the other hand our proof of cut-elimination removes parallelism from strategies, which is not satisfying since this is an essential feature of expansion proofs.

Summary and future work

The next natural steps would be :

- A parallel cut-elimination result for infinitary expansion proofs.
- Extraction of a proof of termination for finitary expansion proofs from this.
- Extensions of this method to fragments of second order arithmetic, trying to understand what they mean from a game-theoretic point of view.

The completion of the two first points would show that expansion proofs are completely adaptable to arithmetic.

Contents

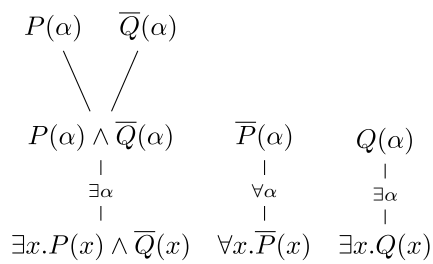

Figure 1: An expansion forest.

1 Background

Here we present informally the concepts required to understand our report.

1.1 Expansion forests

It is well known that intuitionistic logic enjoys a witness property : if $\exists x.A(x)$ is provable, then there exists a term t such that $A(t)$ is provable. This is a straightforward implication of the cut-elimination theorem.

In its most crude form, Herbrand's theorem states that in classical logic, if $\exists x.A(x)$ is provable, then there exists some terms $t_1, \ldots, t_n$ such that $A(t_1) \vee \ldots \vee A(t_n)$ is provable. Therefore it is some kind of weak witness property for classical logic. This theorem can be extended to nested quantifiers, with additional care for the dependencies of those quantifiers. This idea can be used to build a compact representation of proofs in various systems, for example simple type theory [15].

When trying to build a compact representation of proofs, there is a dilemma between :

- Compact proofs : they should be represented by a small number of symbols.
- Easily checkable proofs : it should be possible to check in a small amount of time whether a given object is a proof of a given formula.

Those two requirement are competing against each other. On the one hand the usual sequent calculus provides easily checkable proofs, but they are very redundant and lead to unmanageable size quickly, especially in the cut free case. On the other hand one could represent a proof of a formula simply as a natural number bounding the size of its smallest sequent calculus proof, yielding an obvious checking algorithm. Those two examples show the extreme cases : sequent calculus is full of useless (that is, easy to infer) information, whereas the other example gives proofs uncheckable in practice.

The insight given by Herbrand's theorem on this matter is that a proof in classical logic can be represented as the instantiations of its quantifiers. Given those, it is a matter of propositional classical logic to check if the proof is correct. Expansion forests are a formalisation of this idea. We present this with some examples.

We assume given a signature for first-order logic. In this report we will be concerned only with classical logic, and any proof should be assumed classical. We will use the involution of negation, so we use only $\exists, \forall, \wedge, \vee$ and (possibly negated) atoms. The dual of a formula A is denoted $\overline{A}$ and is defined inductively by $\overline{\exists x.A(x)} = \forall x.\overline{A}(x)$, $\overline{A \vee B} = \overline{A} \wedge \overline{B}$, and so on...

The sequent $\forall x.P(x) \to Q(x), \exists x.P(x) \vdash \exists x.Q(x)$ with predicate symbols P, Q is certainly derivable. In Figure 1 we give an expansion forest proving this.

Note that the formulas at the bottom of this figure are the one-sided representations of the formulas we are trying to prove. These formulas form what is called the *shallow* sequent of the expansion tree, which is the proven sequent. The formulas on the top of the figure are atoms.

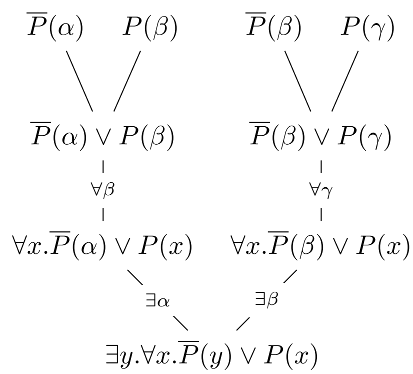

Figure 2: An expansion forest for the drinker's formula.

From them some propositional formulas can be reconstructed, these form the *deep* sequent of the expansion forest. Here the deep sequent is $P(\alpha) \wedge \overline{Q}(\alpha), \overline{P}(\alpha), Q(\alpha)$, note that it is valid.

The edges labelled by terms are called expansions, they carry an order $<$ called the dependency relation, which indicates the order in which the instantiations are performed. By abuse of language, we often identify expansions and their labels. Here we only have $\forall\alpha < \exists\alpha$ for both expansions labeled $\exists\alpha$.

For an expansion forest to be correct, two requirements should be checked :

- The deep sequent should be propositionally valid. This is checkable in coNP.
- The dependency relation should be acyclic. This is checkable in P.

We already see that expansion forests separate two parts in a proof : the propositional part which is kept implicit because it is easily checkable, and the quantifier part which is explicit. Herbrand's theorem can be seen as the statement that this formalism is complete in the sense that if a sequent is provable, then there exists an expansion proof with this sequent as shallow sequent.

An example with a classical flavour is the drinker formula $\vdash \exists y.\forall x.P(y) \to P(x)$ with a predicate symbol P. In Figure 2 we present an expansion forest for it. The deep sequent of this expansion forest is $\overline{P}(\alpha) \vee P(\beta), \overline{P}(\beta) \vee P(\gamma)$ which is valid, and the order on the expansions is linear : $\exists\alpha < \forall\beta < \exists\beta < \forall\gamma$.

What should be noticed here is that the instantiations of $\exists y$ do not take place at the same time. This feature is very specific to classical logic. If we see expansion forests as strategies for games, the player choosing instantiations of existential quantifiers is allowed to do some backtracking, i.e. she can change her mind about a previous move. We will discuss this analogy at length in Section 3.

1.2 Expansion proofs

The formalism for expansion forests sketched in Section 1.1 is only suitable for analytical proofs, i.e. proofs built using only sub-formulas of the formulas being proven. Gentzen's cut-elimination theorem [7] is arguably the most important result in proof theory, and states that analytical proofs are complete for pure first-order logic.

It is very natural to try to extend expansion forests with cuts, and several formalisms have been proposed to do so : proof forests [11], Herbrand nets [14] and expansion proofs [1]. Here we present expansion proofs.

Remark 1. *The terminology on these matters can be a bit confusing. In the literature the formalism sketched in 1.1 is sometimes called expansion trees. In this report an expansion tree is a single tree, and an expansion forest is a family of expansion trees. An expansion pre-proof*

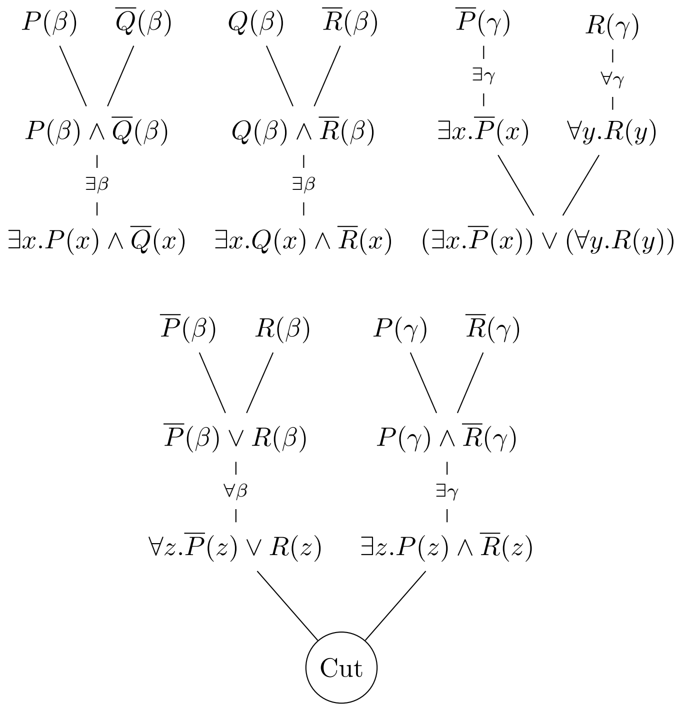

Figure 3: An expansion proof.

is an expansion forest with cuts (represented as indicated below), and it is an expansion proof if it is correct.

Expansion proofs are build from expansion forests by adding *cut node*. A cut node connects an expansion tree for A and another for $\overline{A}$ for some formula A. As an example, we give in Figure 3 an expansion proof with predicate symbols P, Q, R, and with a cut on $\forall z.\overline{P}(z) \vee R(z)$.

This addition makes expansion proofs similar to proof nets [9], and especially to proof nets for classical logic [10, 17, 13]. One difference is that a propositional part is omitted from expansion proofs. Similarly to proof nets, expansion proofs are parallel in the sense that some logical inferences happen at the same time.

It is proven in [1] that expansion proofs are sound and complete with respects to sequent calculus for pure first-order logic, and have a weakly normalising cut-elimination procedure. Our task here is to extend expansion proofs to arithmetic.

1.3 Consistency of Peano arithmetic

Gentzen used sequent calculus to prove consistency of Peano arithmetic [8]. This result looks somehow unsatisfying since it uses a principle stronger than Peano arithmetic (namely induction up to the ordinal number ε_0, which is the smallest fixpoint of $\alpha \mapsto \omega^\alpha$). But by Gödel incompleteness theorem this can not be avoided.

The method of Gentzen was to assign to any hypothetical proof of the empty sequent an ordinal smaller than ε_0, and to define a cut-elimination procedure which strictly decreases the ordinal of a proof. This is provable in Peano arithmetic, and so is the impossibility of a normal proof of the empty sequent. The only principle needed to imply consistency, which goes beyond Peano arithmetic, is the termination of the cut-elimination procedure, i.e. induction up to ε_0.

This proof was later simplified by Schütte [18] using a so called *semi-formal* system, i.e. an infinitary proof system. In this system there is an infinitary deduction rule stating that to prove $\forall x.A(x)$, it is enough to give a proof of $A(n)$ for each natural number n. An exposition of this method can be found in e.g. [20].

$x = x$	$0 + x = x$
$y \neq x, x = y$	$s(x) + y = s(x + y)$
$x \neq y, y \neq z, x = z$	$0 \times x = 0$
$x \neq y, s(x) = s(y)$	$s(x) \times y = y + x \times y$
$x \neq x', y \neq y', x + y = x' + y'$	$0 \neq s(x)$
$x \neq x', y \neq y', x \times y = x' \times y'$	$s(x) \neq s(y), x = y$

Table 1: The system Q^-.

A good formalism for expansion proofs for arithmetic should enable us to prove this result. Indeed it will be proved using infinitary expansion proofs.

2 Expansion proofs with induction

Here we define a generalisation of expansion proofs with a rule for arithmetical induction. It is a relatively straightforward extension of [1]. We are again trying to build proofs meeting two competing requirements : easy checkability and compactness.

The main idea is that a proof in Peano Arithmetic can be given by :

- The formulas on which induction is performed.
- The instantiations of first-order quantifiers, both in the formulas being proven and in the formulas introduced by the inductions.

It is well known that Peano arithmetic does not enjoy cut-elimination (e.g. Theorem 10.4.12 in [22]). This is a precise formulation of the fact that in order to prove A, one sometimes need to perform an induction on a formula B which is not a subformula of A.

In this section will define expansion proofs with induction, and then we will sketch the proof that they are sound and complete with respect to PA. Then we will examine some cut-elimination procedure for them. We did not succeed in proving termination of this procedure, instead a cut-elimination result will be obtained using infinitary expansion proofs in Section 3.

We will call expansion proofs with induction simply expansion proofs in the rest of this report, since there is no risk of confusion : we will only consider expansion proofs with induction.

Note that $A, B, C, \ldots$ are used to denote arbitrary formulas. The notation $\exists x.A(x)$ with A a formula is convenient because we denote the substitution of x by α in A by $A(\alpha)$. This does not mean that x occurs in A, or that x is its only free variable.

2.1 Definition

Here we define expansion proofs.

2.1.1 Q^-

We assume given a signature $0, s, +, \times$ for arithmetic. We assume given the set Q^- of axioms for arithmetic, containing the universal closure of the formulas presented in Table 1.

We do not know whether $Q^- \vdash A_1, \ldots, A_n$ for $A_1, \ldots, A_n$ quantifier free (but not variable free !) is decidable. Our presentation will assume that it is not, but we will explain what can be simplified if it turns out to be decidable.

2.1.2 Expansion trees

Definition 1. *Inductively, we define the set of expansion trees together with a function* Sh *mapping expansion trees to formulas :*

- *$s = t$ is an expansion tree with* $\mathrm{Sh}(s = t) = s = t$.
- *If E_i are expansion trees with* $\mathrm{Sh}(E_i) = A_i$ *for $i \in \{1, 2\}$, then for $\circ \in \{\wedge, \vee\}$, $E_1 \circ E_2$ is an expansion tree with* $\mathrm{Sh}(E_1 \circ E_2) = \mathrm{Sh}(E_1) \circ \mathrm{Sh}(E_2)$.
- *If $E_1, \ldots, E_n$ are expansion trees with* $\mathrm{Sh}(E_i) = A(t_i)$*, then $\exists x.A(x) +^{t_1} E_1 + \ldots +^{t_n} E_n$ is an expansion tree with* $\mathrm{Sh}(\exists x.A(x) +^{t_1} E_1 + \ldots +^{t_n} E_n) = \exists x.A(x)$.
- *If E is an expansion tree with* $\mathrm{Sh}(E) = A(y)$*, then $\forall x.A(x) +^y E$ is an expansion tree with* $\mathrm{Sh}(\forall x.A(x) +^y E) = \forall x.A(x)$.

For an expansion tree E we call $\mathrm{Sh}(E)$ *the shallow formula of E.*

We call any $+^t$ an expansion. We denote it $\forall\alpha$ if it is in $\forall x.A +^{\alpha} E$ (and we say that $\forall\alpha$ dominates any expansion in E) and $\exists t$ if it is in $\exists x.A + \ldots +^t E + \ldots$ (and we say that $\exists t$ dominates any expansion in E) . Moreover a variable α in $\forall x.A +^{\alpha} E$ is called an eigenvariable.

Note that $n = 0$ in the third case is allowed, so $\exists x.A(x)$ is an expansion tree.

This corresponds to the graphical notation used in Section 1 using the following translation.

Definition 2. *We define inductively the graphical representation* $\mathrm{G}(E)$ *of an expansion tree E*

- $\mathrm{G}(s = t)$ *is* $s = t$.
- $\mathrm{G}(E_1 \circ E_2)$ *is*
$$\begin{array}{ccc} \mathrm{G}(E_1) & & \mathrm{G}(E_2) \\ & \diagdown \quad \diagup & \\ & \mathrm{Sh}(E_1) \circ \mathrm{Sh}(E_2) & \end{array}$$
- $\mathrm{G}(\exists x.A(x) +^{t_1} E_1 + \ldots +^{t_n} E_n)$ *is*
$$\begin{array}{ccc} \mathrm{G}(E_1) & \cdots & \mathrm{G}(E_n) \\ \diagdown & | & \diagup \\ \exists t_1 & \cdots & \exists t_n \\ \diagdown & | & \diagup \\ & \exists x.A(x) & \end{array}$$
- $\mathrm{G}(\forall x.A(x) +^y E)$ *is*
$$\begin{array}{c} \mathrm{G}(E) \\ | \\ \forall y \\ | \\ \forall x.A(x) \end{array}$$

Intuitively, an expansion tree with shallow formula A is a sequence of inferences performed on A. The existential expansions can be seen as moves by the *prover*, and universal expansions as moves by the *disprover*. This intuitive game-theoretic point of view will be made precise in Section 3 where we present infinitary expansion proofs.

2.1.3 Expansion pre-proofs

A formula is said positive if it begins with $\exists, \vee$ or it is a non-negated atom.

Definition 3. *A cut on a formula A is an ordered pair of expansion tree (E_1, E_2) with* $\mathrm{Sh}(E_1) = \overline{\mathrm{Sh}(E_2)} = A$ *and* $\mathrm{Sh}(E_1)$ *positive.*

We choose $\mathrm{Sh}(E_1)$ positive arbitrarily for the sake of canonicity of notations. We say that a cut (E_1, E_2) dominates the expansions in E_1 and E_2.

Definition 4. *An induction on a formula $A(t)$ (with some occurrences of t indicated) is a triple (E_0, E_s, E) with* $\mathrm{Sh}(E_0) = A(0)$, $\mathrm{Sh}(E_s) = \forall x.\overline{A}(x) \vee A(s(x))$ *and* $\mathrm{Sh}(E) = \overline{A}(t)$.

We say that an induction (E_0, E_s, E) dominates the expansions in E_0, E_s and E. These inductions model a cut followed by an induction in sequent calculus :

$$
\dfrac{\dfrac{\dfrac{\dots}{\Gamma, A(0)} \qquad \dfrac{\dots}{\Gamma, \forall x.\overline{A}(x) \vee A(s(x))}}{\Gamma, A(t)} \qquad \dfrac{\dots}{\Gamma, \overline{A}(t)}}{\Gamma}
$$

It is natural to consider directly those since the other cuts can be eliminated as for pure first-order logic. This will be proven for expansion proofs in Section A.3. A similar rule was considered for example in [3], in the more general context of inductively defined predicates.

For a set $\mathcal{P}$ of expansion trees, cuts and inductions :

$$E_1, \dots, E_n, (F_1, G_1), \dots, (F_m, G_m), (H_1^0, H_1^s, H_1), \dots, (H_l^0, H_l^s, H_l)$$

we define $\mathrm{Sh}(\mathcal{P}) = \mathrm{Sh}(E_1), \dots, \mathrm{Sh}(E_n)$. This corresponds to the sequent being proven by $\mathcal{P}$ so it is natural to discard formulas introduced by cuts.

Definition 5. *An expansion pre-proof is a finite set $\mathcal{P}$ of expansion trees, cuts and inductions such that :*

- *Any two eigenvariables of $\mathcal{P}$ are distinct.*
- *There is no eigenvariable of $\mathcal{P}$ in $\mathrm{Sh}(\mathcal{P})$.*

These two side conditions are technical details. One thing which should be noted is that eigenvariables are allowed to occur in the shallow formulas of cuts and inductions.

2.1.4 Expansion proofs

In this section we present expansion proofs.

Definition 6. *We define* Dp *on any expansion tree by :*

- $\mathrm{Dp}(s = t) = s = t$
- $\mathrm{Dp}(E_1 \circ E_2) = \mathrm{Dp}(E_1) \circ \mathrm{Dp}(E_2)$ *where* $\circ \in \{\wedge, \vee\}$
- $\mathrm{Dp}(\exists x.A(x) +^{t_1} E_1 \dots +^{t_n} E_n) = \bigvee_{i=1}^{n} \mathrm{Dp}(E_i)$
- $\mathrm{Dp}(\forall x.A(x) +^{y} E) = \mathrm{Dp}(E)$

We extend it to cuts and inductions by

- $\mathrm{Dp}((E_1, E_2)) = \mathrm{Dp}(E_1) \wedge \mathrm{Dp}(E_2)$
- $\mathrm{Dp}((E_0, E_S, E)) = \mathrm{Dp}(E_0) \wedge \mathrm{Dp}(E_s) \wedge \mathrm{Dp}(E)$

And finally for an expansion pre-proof $\mathcal{P} = E_1, \dots, E_n$ we define $\mathrm{Dp}(\mathcal{P}) = \mathrm{Dp}(E_1), \dots, \mathrm{Dp}(E_n)$.

Note for the third case that the empty disjunction is false. We call $\mathrm{Dp}(\mathcal{P})$ the deep sequent of $\mathcal{P}$. One condition for $\mathcal{P}$ to be an expansion proof will be that $Q^- \vdash \mathrm{Dp}(\mathcal{P})$. The second condition will impose some restrictions on the expansions, cuts and inductions in $\mathcal{P}$.

Definition 7. *We call the expansions, cuts and inductions of an expansion pre-proof $\mathcal{P}$ its events.*

Definition 8. *Let $\mathcal{P}$ be an extension pre-proof. We define a relation $<_{\mathcal{P}}$ on its events as the smallest relation such that :*

- *$q <_{\mathcal{P}} q'$ if q dominates q'.*

- $\forall\alpha <_{\mathcal{P}} \exists t$ *if* α *occurs in* t
- $\forall\alpha <_{\mathcal{P}} (E_1, E_2)$ *if* α *occurs in* $\mathrm{Sh}(E_1)$
- $\forall\alpha <_{\mathcal{P}} (E_0, E_s, E)$ *if* α *occurs in* $\mathrm{Sh}(E)$

This relation on events is called the dependency relation. It indicates roughly the order in which inferences are performed, more precisely some constraints on this order which cannot be violated.

Definition 9. *An expansion proof is composed of :*

- *An expansion pre-proof* $\mathcal{P}$ *such that* $<_{\mathcal{P}}$ *is acyclic.*
- *A proof of* $Q^- \vdash \mathrm{Dp}(\mathcal{P})$*.*

Note that the criterion on $<_{\mathcal{P}}$ is *global*, as opposed to the criteria for sequent calculus which are *local*.

The proof of $Q^- \vdash \mathrm{Dp}(\mathcal{P})$ can be represented in sequent calculus or in any suitable system for classical first-order logic (for example expansion proofs without inductions !). The presence of this additional proof is here to guarantee the decidability of whether a given object is an expansion proofs. If $Q^- \vdash A_1, \ldots, A_n$ with $A_1, \ldots A_n$ quantifier free turns out to be decidable, then we do not need to ask for the proof of $Q^- \vdash \mathrm{Dp}(\mathcal{P})$: we only ask for its existence.

We will often refer to an expansion proof as its underlying expansion pre-proof $\mathcal{P}$ in the rest of this report. Moreover when we say that an expansion pre-proof $\mathcal{P}$ is an expansion proof, we mean that we have a canonical way to build a proof of $Q^- \vdash \mathrm{Dp}(\mathcal{P})$.

The acyclicity of $<_{\mathcal{P}}$ is very natural, it states that the order in which inferences are performed is not cyclic, i.e. the constraints on this order are not contradictory. It will be necessary in order to sequentialise an expansion proof into a sequent proof.

From a game point of view the dependency relation indicates some constraints on the order in which the moves are performed by the two players.

2.2 Summary of the properties of expansion proofs

Here we present informally the properties of expansion proofs with induction. A precise account can be found in Appendix A.

2.2.1 Soundness and completeness with respect to Peano arithmetic

In Section A.2 we define a presentation of Peano arithmetic (shortened to PA from now on) in a one sided sequent calculus and we show :

Theorem 1. *There exists a proof of* Γ *in PA if and only if there exists an expansion proof* $\mathcal{P}$ *with* $\mathrm{Sh}(\mathcal{P}) = \Gamma$*.*

For the proof of completeness we inductively translate proofs in PA into expansion proofs. The crucial point is that sometimes we need to build a proof of Γ out of two. As an example when dealing with :

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \wedge$$

by induction hypothesis we have two expansion proofs $\mathcal{P}, E_A$ with $\mathrm{Sh}(\mathcal{P}, E_A) = \Gamma, A$ and $\mathcal{P}', E_B$ with $\mathrm{Sh}(\mathcal{P}', E_B) = \Gamma, B$. We should use $E_A \wedge E_B$, but we need to have a unique expansion proof $\mathcal{P}''$ with $\mathrm{Sh}(\mathcal{P}'') = \Gamma$, which should depend on both $\mathcal{P}$ and $\mathcal{P}'$. In order to deal with this we need to introduce an operation called merge.

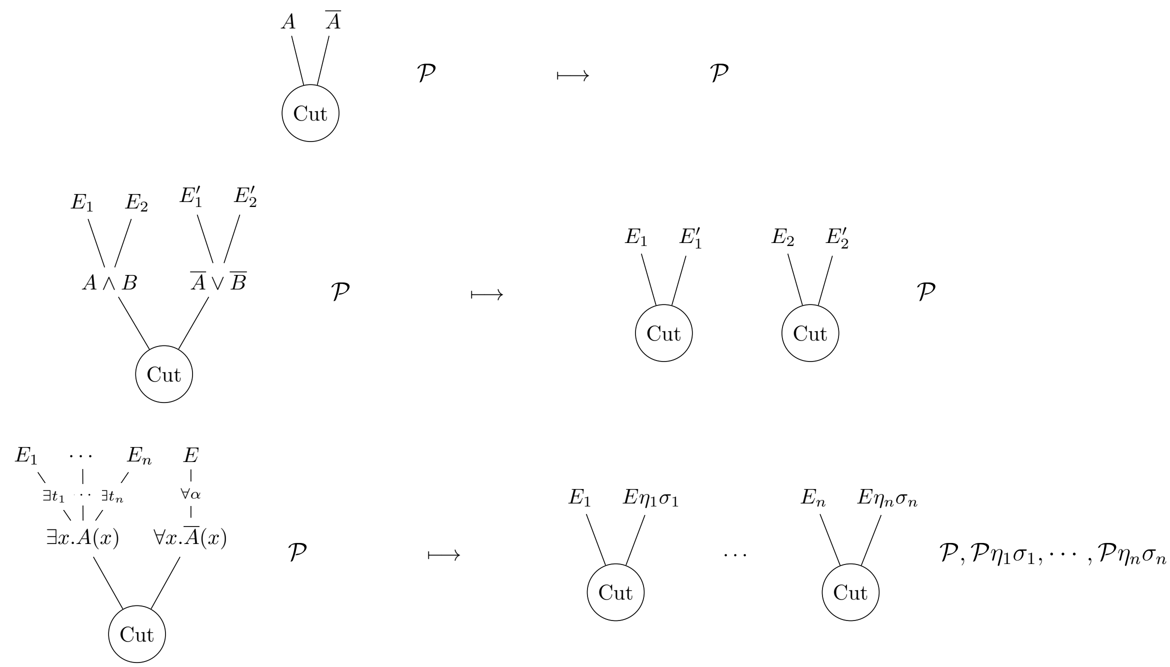

Figure 4: Elimination of cuts

This merge operation (unsurprisingly) is able to merge two different expansion trees or two cuts with the same shallow formula. It plays a role similar to contractions in the sense that it can transform a proof of say Γ, A, A into a proof of Γ, A.

More precisely, from a sequent calculus perspective it is introducing contractions at the bottom of the proof, and then pushing them upward in the proof, until there is only contractions on existential formulas and atoms left. Contractions on existential formulas are allowed in expansion proofs. Contractions on atoms can be erased from expansion proofs since they can be performed during the checking of the deep sequent. So the merge operation builds an expansion proof.

Moreover the merge operation is able to identify different eigenvariables which corresponds to the same contracted universal quantifier. This is presented in details in Section A.1.2.

In the other direction, in order to translate an expansion proof to a sequent calculus proof we need to sequentialise it. This is done by translating in priority the events minimal with respect to the dependency relation.

2.2.2 About cut-elimination

It is possible to define a cut-elimination procedure directly on expansion proofs.

First we define some rewriting rules.

Definition 10. *We define a rewriting system on expansion proofs in Figure 4.*

The first rule can be applied only when A is atomic.

In the third rule, the η_i are renaming eigenvariables and $\sigma_i(\alpha) = t_i$.

It is not surprising that we need to duplicate $\mathcal{P}$ in the case of a $\exists/\forall$ cut since $\exists$ nodes are (implicit) contractions. Note that we need to keep an untouched copy $\mathcal{P}$. A similar phenomena appears in the behaviour of ε-calculus [16].

There is one obvious problem with these rewriting rules : the shallow sequent is not preserved by the reduction procedure. Indeed an expansion proof for Γ can be transformed into one for Γ, Γ. Another subtler problem is that these reductions are not normalising, because the reduction for

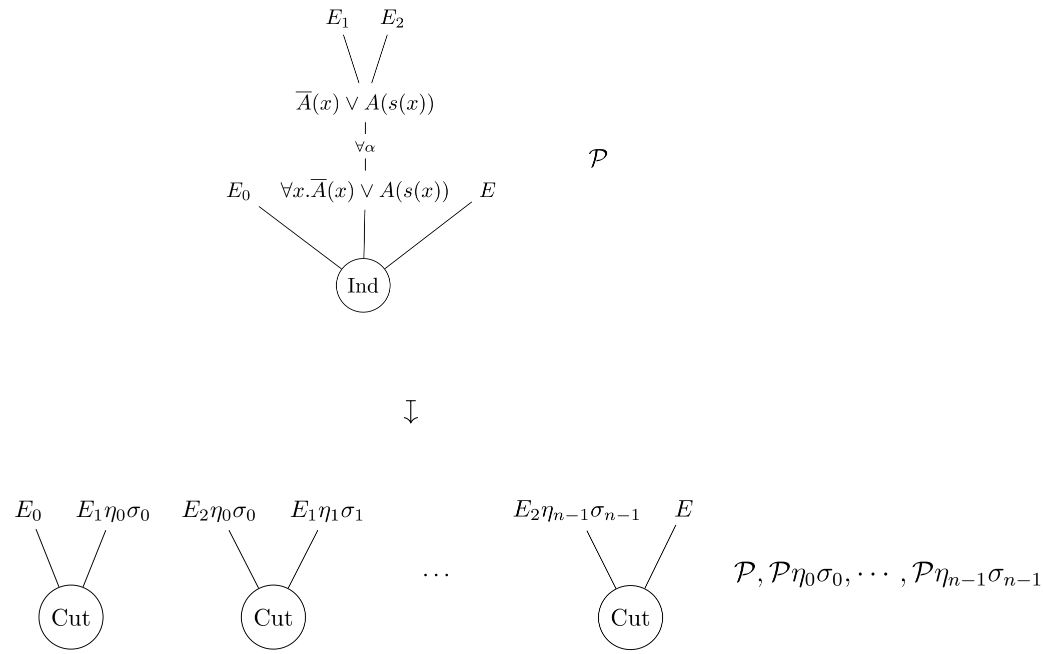

Figure 5: Elimination of closed inductions

cuts on quantified formulas is duplicating too much. We need to keep these duplications under control. These two problems can be solved by the merge construction, if we care for merging multiple cuts on the same formula.

It is shown in section A.3 that these rules (composed with the suitable merges) are weakly normalising. But they do not eliminate inductions, and therefore this result does not entail consistency of PA.

We can try to eliminate induction on closed terms by adding the rule displayed in Figure 5 when $\mathrm{Sh}(E) = A(s^n(0))$. In this figure, the η_i are renaming the eigenvariables to fresh variables and $\sigma_i(\alpha) = s^i(0)$. Note that here again there is an untouched copy of $\mathcal{P}$. This can be adapted for $A(t)$ with any closed term t, since any closed term is provably in Q^- equal to $s^n(0)$ for some natural number n.

Of course it should be composed with merges. It is shown correct in Section A.3.5.

The normalisation of the cut-elimination rules of Definition 10 together with closed induction elimination would imply consistency of PA. Unfortunately we did not manage to prove this normalisation result. Instead we will use infinitary expansion proofs.

3 Infinitary expansion proofs for arithmetic

The consistency proof of PA by Gentzen [8] was later simplified using an infinitary proof system [18]. We will adapt these methods to expansion proofs, by defining infinitary expansion proofs.

We obtain a cut-elimination procedure for infinitary expansion proofs. This result is somehow unsatisfying since the procedure is heavily non-confluent. This will be discussed at length later.

Infinitary expansion proofs are also a natural way to make the game interpretation of expansion proofs for arithmetic precise. Indeed infinitary expansion proofs are necessary in order to get rid of inductions, which are not very natural from a game perspective. To emphasise this connection, infinitary expansion pre-proofs are called strategies and infinitary expansion proofs are called winning strategies. These strategies are inspired by Coquand's arithmetical games [6].

3.1 Definition of strategies

In this section we define strategies.

3.1.1 Infinitary formulas

From now on we will use infinitary formulas.

Definition 11. *We define the formulas inductively.*

- $\perp$ *and* $\top$ *are formulas.*
- *If for every natural number* i*,* A_i *is a formula, so are* $\bigvee_i A_i$ *and* $\bigwedge_i A_i$*.*

Any infinitary formula A has a natural dual $\overline{A}$, corresponding to its negation. An arithmetical sentence can be interpreted by an infinitary formula.

Definition 12. *Let* A *be an arithmetical sentence. We inductively define an infinitary formula* A^∞ *:*

- $(s = t)^\infty$ *is* $\top$ *or* $\perp$ *depending on whether* $s = t$ *is true (this is equivalent to provability in* Q^- *since we are dealing with closed formulas).*
- $(A \wedge B)^\infty = \bigwedge_i S_i$ *with* $S_0 = A^\infty$*,* $S_1 = B^\infty$ *and* $S_i = \top$ *otherwise.*
- $(A \vee B)^\infty = \bigvee_i S_i$ *with* $S_0 = A^\infty$*,* $S_1 = B^\infty$ *and* $S_i = \perp$ *otherwise.*
- $(\forall x.A(x))^\infty = \bigwedge_i (A(i))^\infty$.
- $(\exists x.A(x))^\infty = \bigvee_i (A(i))^\infty$.

Using this translation, we will sometimes use e.g. the infinitary formula $A \wedge B$ where A and B are infinitary formulas. This denotes $\bigwedge_k A_k$ with $A_0 = A$, $A_1 = B$ and $A_k = \top$ otherwise.

In Section B.2.1 we define LK_∞, which is the basic sequent-calculus for infinitary formulas. An arithmetical formula A such that LK_∞ proves A^∞ is called ω-provable.

An arithmetical formula is ω-provable if and only if it is true interpreted in the natural numbers. This is not surprising : the definition of ω-provability is very similar to the Tarski's truth definition [21]. So provability in this system is highly undecidable.

3.1.2 The game-theoretic interpretation

Here we present intuitively the game-theoretic interpretation of strategies, since it enlightens the next definitions of this section.

The existential $\bigvee$ (resp. universal $\bigwedge$) quantifiers are instantiated by a player which will be called Eve (resp. Adam).

Intuitively, a strategy for a sequent Γ indicates a possible strategy for Eve in the game represented by Γ.

We will define a rewriting of strategies corresponding to moves in the game. Our point of view in that a move in the game creates a new game (the old game after the move), and that this reduction of strategies extracts the strategy for the new game from the old strategy. There will be two kinds of rewritings :

- The universal moves corresponds to Adam choosing which of the A_i in $\bigwedge_i A_i$ Eve has to prove.
- The existential moves corresponds to Eve trying to prove an A_i in $\bigvee_i A_i$. These moves do not discard the other choices, indeed Eve can change her mind later.

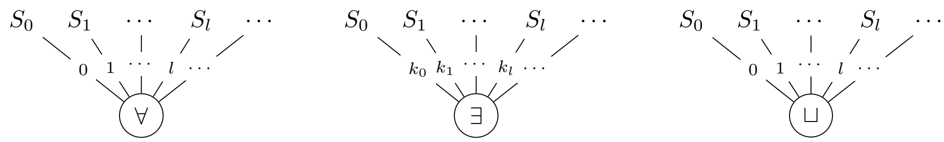

Figure 6: The graphical representation of strategies

The idea is that every move by Eve is depending on a finite number of moves by Adam, and Eve will play it if and only if Adam has previously played all these moves.

These rewriting rules will be strongly normalising, meaning that every play has a finite number of moves. We will say that a strategy is winning if every normal forms of the strategy are playing a game corresponding to $\top$. This means that whatever moves are performed (in whichever order), Eve eventually wins.

Note that our games are far from alternating. Moreover there will be a merge construct denoted $\sqcup$ which will allow Eve to influence the order of the moves.

3.1.3 Structures

Structures are the building blocks of strategies, in the same way that expansion trees are the building blocks of expansion proofs.

Definition 13. *We inductively define structures for infinitary formulas.*

- $\top$ *is a structure for* $\top$*.*
- $\bot$ *is a structure for any formula.*
- *If for all natural numbers* i*, we have a structure* S_i *for* A_i*, then* $\forall_{i\in\omega} i.S_i$ *is a structure for* $\bigwedge_i A_i$*.*
- *Let* $I \subseteq \omega$*. If for all* $i \in I$ *we have a natural number* k_i *and a structure* S_i *for* A_{k_i} *then* $\exists_{i\in I} k_i.S_i$ *is a structure for* $\bigvee_i A_i$*.*
- *If for all natural numbers* i *we have a structure* S_i *for* A*, then* $\sqcup_{i\in\omega} i.S_i$ *is a structure for* A*.*

Graphically we represent $\forall_{i\in\omega} i.S_i$, $\exists_{i\in I} k_i.S_i$ and $\sqcup_{i\in\omega} i.S_i$ as indicated in Figure 6.

These structures can be represented by trees, i.e. sets of finite sequences of natural numbers closed under prefix. Those trees are well-founded in the sense that they have no infinite branch. An edge above an $\exists$ node (resp. $\forall$ node) will be called an existential (resp. universal) edge. An edge above a $\sqcup$ node is called an $\sqcup$ edge. By abuse of language, we will identify an edge and its label when there is no risk of confusion.

We say that an edge labelled l (or k_l) in Figure 6 *dominates* the edges in the structure S_l above it.

The fact that $\bot$ is a strategy for any formula corresponds to a weakening rule : Eve has a default strategy for any formula, which is loosing.

The $\sqcup_{k\in\omega} k.S_k$ construct will be linked to a universal node, indicating that if Adam plays k for this universal node, then S_k will be played. This will be mostly used by Eve in order to force the moves to happen is some order.

3.1.4 Strategies

Strategies are infinitary expansion pre-proofs. First we define structures meant to represent cuts.

Definition 14. *Let* A *be a formula. A cut-structure for* A *is a structure for* $A \wedge \overline{A}$*.*

We say that a cut-structure *dominates* the edges in it.

Definition 15. *Let $A_1, \ldots, A_n$ be formulas. A structure forest for $A_1, \ldots, A_n$ is :*

- *A structure for each A_i*
- *A countable set of cut-structures for any formula.*

The cuts are played by Eve, they challenge Adam to play one game of his choice among A and $\overline{A}$.

Definition 16. *Let S be a structure forest, we call the cut-structures and existential edges in S the events of S.*

A function Dep *which associates to any event of S a finite set of universal edges in S is called a dependency relation for S.*

A function Lbl *which associates to any $\sqcup$ node a $\forall$ node is called a labelling of S. The k-th edge above a $\sqcup$ node n is said to correspond to the k-th universal node above* $\mathrm{Lbl}(n)$.

A structure forest together with a dependency relation and a labelling is called a pre-strategy.

Note the definition for event is different than in the finitary case : we did not include the universal edges.

The Dep function is associating to any possible move of Eve a set of moves by Adam. Eve will play a move s if and only if all the moves in $\mathrm{Dep}(s)$ have been played by Adam.

If $\sqcup$ in $\sqcup_{k\in\omega}k.S_k$ is labelled by a $\forall$ node s, then S_k will be played if and only if Adam plays k at s.

Definition 17. *A pre-strategy obeys the finite reaction principle if for any finite set S of universal edges, there is only a finite number of events s such that* $\mathrm{Dep}(s) = S$.

The finite reaction principle states that a finite number of moves by Adam can only trigger a finite number of moves by Eve.

Definition 18. *Let S be a pre-strategy. We define a relation $<_S$ on events, $\sqcup$-edges and universal edges as the smallest relation such that :*

- *$s <_S t$ if s dominates t.*
- *$s <_S t$ if t is an event and* $s \in \mathrm{Dep}(t)$.
- *$s <_S t$ if t is a $\sqcup$ edge corresponding to the universal edge s.*

Definition 19. *A pre-strategy S is called a strategy if :*

- *S obeys the finite reaction principle.*
- *$<_S$ has no infinite increasing sequence.*

The second condition is similar to the acyclicity condition for expansion proofs. Indeed it implies that $<_S$ has no cycles. It will be crucial in the proof that any play between Adam and Eve is finite.

3.1.5 Winning strategies

Definition 20. *Let S be a strategy and e_l be a universal edge labelled by l. The operation $-_{e_l}$ is defined inductively on each structure :*

- $-_{e_l}(\forall_{i\in\omega}i.S_i) = \forall_{i\in\omega}i. -_{e_l}(S_i)$.
- $-_{e_l}(\exists_{i\in I}k_i.S_i) = \exists_{i\in I}k_i. -_{e_l}(S_i)$ *with e_l removed from* $\mathrm{Dep}(k_i)$.
- $-_{e_l}(\sqcup_{i\in\omega}i.S_i) = S_l$ *if $\sqcup$ corresponds to the node of e_l.*

- $-_{e_l}(\sqcup_{i\in\omega} i.S_i) = \sqcup_{i\in\omega} i. -_{e_l}(S_i)$ *otherwise.*

The strategy $\{-_{e_l}(T) \mid T \in S\}$ *is denoted* $-_{e_l}(S)$.

Definition 21. *We define some reduction rules denoted by* $\mapsto$ *and called moves on strategies.*

If $\forall_{i\in\omega} i.S_i$ *is a structure we denote* e_i *the universal edge labelled by* i *in* $\forall_{i\in\omega} i.S_i$. *For any natural number* l *:*

$$S, \forall_{i\in\omega} i.S_i \mapsto -_{e_l}(S, S_l)$$

If $\forall_{i\in\omega} i.S_i$ *is a cut-structure we denote* e_i *the universal edge labelled by* i *in* $\forall_{i\in\omega} i.S_i$. *If* $\mathrm{Dep}(\forall_{i\in\omega} i.S_i) = \emptyset$, *for any natural number* l *:*

$$S, \forall_{i\in\omega} i.S_i \mapsto -_{e_l}(S, S_l)$$

If $l \in I$, *we denote* e_l *existential edge labelled by* k_l *in* $\exists_{i\in I} k_i.S_i$. *If* $\mathrm{Dep}(e_l) = \emptyset$, *then*

$$S, \exists_{i\in I} k_i.S_i \mapsto S, \exists_{i\in I-\{l\}} k_i.S_i, S_l$$

The first two reduction corresponds to a play by Adam, the last reduction to a play by Eve. The conditions on Dep are a precise definition of what we meant by : Eve can play s only if all the moves in $\mathrm{Dep}(s)$ have been played before by Adam.

In Section B.1.2, we are able to prove :

Lemma 1. *The reduction* $\mapsto$ *is strongly normalising.*

To show this we extract an infinite increasing sequence for $<_S$ from an infinite reduction.

Definition 22. *We call a play for* S *a sequence of reduction for* $\mapsto$ *from* S, *with the last reduct being a normal form.*

With this terminology, Lemma 1 says that any play between Eve and Adam is finite. The next lemma is immediate.

Lemma 2. *A normal form for* $\mapsto$ *contains only :*

- *Structures of the form* $\top$, $\bot$.
- *Structures of the form* $\exists_{i\in I} k_i.S_i$ *s.t. for all* i, $\mathrm{Dep}(k_i) \neq \emptyset$.
- *Structures of the form* $\sqcup_{i\in\omega} i.S_i$.
- *Cut-structures* C *with* $\mathrm{Dep}(C) \neq \emptyset$

Definition 23. *We say that a play is winning if the normal form at the end of the sequence contains* $\top$.

Definition 24. *A strategy is winning if all its plays are winning.*

It is straightforward from this definition that $\mapsto$ maps winning strategies to winning strategies.

3.2 Properties of strategies

Here we summarise the properties of strategies. A precise account can be found in Appendix B.

3.2.1 The ordinal of a strategy

In order to prove results on strategies, we need to have some measure of their complexities. Recall that any play in a strategy S is finite, but since there is often an infinite number of possible next moves (if and only if Adam can play), we cannot define the maximal length of a play as a measure.

For any family of ordinals $(\alpha_i)_{i\in I}$, we denote $\sup_i(\alpha_i+1)$ by $\sup_i^*(\alpha_i)$. We recall the height of a well founded-tree, which is an ordinal.

Definition 25. *The height of a well-founded tree is defined by :*

- *The height of a leaf is* 0*.*
- *If the set I of sons of a node s have heights $(\alpha_i)_{i\in I}$, then the height of s is $\sup_i^*(\alpha_i)$.*

The height of a tree is the height of its root.

The use of $\sup^*$ guarantee that the height of a node is strictly bigger than the heights of its sons.

Definition 26. *The ordinal of a strategy S is the height of its tree of moves.*

So the ordinals of strategies are a priori only bounded by ω_1, the smallest non-denumerable ordinal.

We also want to measure how far from cut-free a strategy is, since we want to prove a cut-elimination result. We define the depth of a formula as expected.

Definition 27. *The depth of a formula A is defined inductively.*

- *The depth of $\top$ or $\perp$ is* 0
- *The depth of $\bigwedge_i A_i$ or $\bigvee_i A_i$ is $\sup_i^*(\alpha_i)$ with α_i the depth of A_i.*

Definition 28. *We write $\vdash^\beta_\alpha \Gamma$ to say there exists a strategy S for Γ with ordinal $\leq \alpha$ and such that the formulas of the cuts in S are of depth $< \beta$.*

3.2.2 Composition of strategies

Now we state a few very useful lemmas. They are proven in Section B.1.3.

Lemma 3. *If $\vdash^\beta_\alpha \Gamma, \bigvee_n A_n, A_k$ then $\vdash^\beta_{\alpha+1} \Gamma, \bigvee_n A_n$.*

This is proven by attaching the structure for A_k to the structure for $\bigvee_n A_n$.

Lemma 4. *If for all natural numbers i we have $\vdash^\beta_{\alpha_i} \Gamma, A_i$, then $\vdash^\beta_{\sup_i^* \alpha_i} \Gamma, \bigwedge_n A_n$.*

This is proven by defining the strategy in which Eve asks for Adam to play an i in $\bigwedge_i A_i$ first and then plays according to the given strategy for Γ, A_i. This is somehow unparallelising the proof by forcing $\bigwedge_i A_i$ to be played first. This is where the construction $\sqcup$ is useful.

Lemma 5. *Assume $\vdash^\beta_\alpha \Gamma, A$ and $\vdash^\beta_{\alpha'} \Gamma, \overline{A}$, let us denote the depth of A by δ. Then $\vdash^{\sup(\beta,\delta+1)}_{\sup(\alpha,\alpha')+1} \Gamma$.*

This is a consequence of the previous lemma.

3.2.3 Correctness and completeness with respect to LK_∞

In Section B.2 we prove :

Theorem 2. *A sequent Γ is provable in LK_∞ if and only if there exists a winning strategy for Γ.*

The translation of a proof in LK_∞ into a winning strategy is straightforward using Lemmas 3,4 and 5. The translation of a winning strategy into a proof in LK_∞ is done by induction on the ordinal of the strategy.

An arithmetical sequent Γ is called ω-provable if there is a proof of Γ^∞ in LK_∞. Similarly we say that there exists a winning ω-strategy for Γ if there exists a winning strategy for Γ^∞.

Theorem 3. *An arithmetical sequent Γ is ω-provable if and only if there exists a winning ω-strategy for Γ.*

Proof. This is an immediate consequence of Theorem 2. □

3.2.4 Cut-elimination for strategies

In Section B.4 we prove that if there exists a winning strategy for Γ, then there exists a winning strategy without cut for Γ. From a game point of view this means that the possibility for Eve to force Adam to play either A or $\overline{A}$ for any formula A is useless. So we restore a weak form of analyticity, i.e. Eve has no need to come up with new formulas.

As an example it is possible to prove in PA (with a function symbol f, and the order $<$ suitably axiomatised) that $\exists x.\forall y.f(y) \geq f(x)$. This is done for example using an induction on α in $\forall x.f(x) > \alpha \vee (\exists x.\forall y.f(y) \geq f(x))$, and then instantiating α as $f(0)$.

The cut-elimination result states that this proof can be translated to a winning strategy for Eve in the game $\exists x.\forall y.f(y) \geq f(x)$. In this game, when cuts are not allowed, Eve is allowed to play any natural number m, and then Adam has to find a natural number n such that $f(n) < f(m)$. The winning strategy extracted from the proof will be :

- Eve plays 0.
- Adam tries to play m such that $f(m) < f(0)$.
- Eve changes her mind and play m.
- Adam tries to play m' such that $f(m') < f(m)$.
- Eve changes her mind again, and repeats this process similarly $f(0)$ times.

In the end Eve wins because Adam can not build a strictly decreasing sequence $f(0) > f(m) > \ldots$ of length more than $f(0)$.

An important observation here is that we can not bound a priori the number of times Eve will change her mind, because it depends on $f(0)$. In pure first-order logic, this can not happen, we can bound a priori the number of times Eve will change her mind for a given expansion proof. This phenomena is an essential feature of arithmetic, and justifies the definition of infinitary expansion proofs.

Using Lemmas 3, 4 and 5, we are able to adapt straightforwardly the usual proof of normalisation for LK_∞. In Section B.3.1 we present the cut-elimination result needed for consistency of PA, which is obtained using induction up to ε_0 and elementary means only. In Section B.4 we present a full cut-elimination result, which uses induction up to ω_1.

The only precise result we state here is this theorem from Section B.3.1, which is using induction up to ε_0 only.

Theorem 4. *Assume $\vdash^k_\alpha \Gamma$ with $\alpha < \varepsilon_0$ and k a natural number, then $\vdash^0_{\omega_k(\alpha)} \Gamma$ with $\omega_0(\alpha) = \alpha$ and $\omega_{k+1}(\alpha) = \omega^{\omega_k(\alpha)}$.*

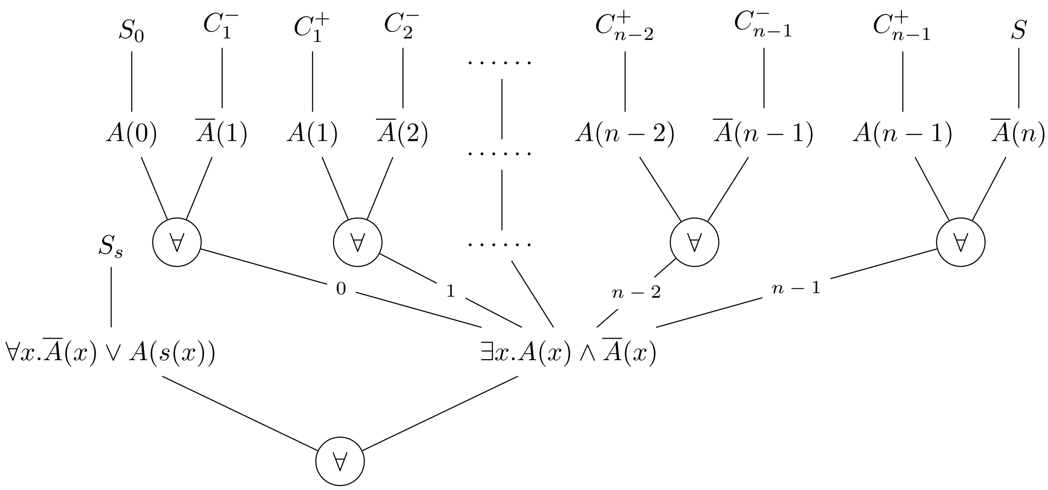

Figure 7: Translation of an induction

3.3 Translation of expansion proofs with induction to infinitary expansion proofs

In this section we sketch the link between expansion proofs and strategies, and prove consistency of PA.

3.3.1 The translation

Using Lemmas 3, 4 and 5, we can show by induction that an expansion proof can be translated to a strategy. In fact an expansion proof for Γ with free variables $\alpha_1, \ldots, \alpha_n$ is translated into a strategy for $\Gamma[\alpha_1/k_1, \ldots, \alpha_n/k_n]$ for any natural numbers $k_1, \ldots k_n$. So we can assume Γ closed. The only case which is not straightforward is the case of inductions.

Assume $\mathcal{P}, (E_0, E_s, E)$ is an expansion proof with $\mathrm{Sh}(\mathcal{P}) = \Gamma$ and $\mathrm{Sh}(E) = A(t)$. We can assume that t is $s^n(0)$. Then by induction we have :

- A strategy for $\Gamma, A(0)$ denoted P_0, S_0.
- A strategy for $\Gamma, \forall x.\overline{A}(x) \vee A(s(x))$ denoted P_s, S_s.
- A strategy for $\Gamma, \overline{A}(n)$ denoted P, S.

We can use the strategy which cuts on $\forall x.\overline{A}(x) \vee A(s(x))$ described in Figure 7. In this figure, C_i^+ and C_i^- are copycat strategies : Eve plays the same as Adam. Moreover P_0, P_s and P are activated according to what Adam plays in this cut, so that Eve win.

In Section B.3.2 this is presented precisely, in order to prove :

Lemma 6. *Let Γ be a closed arithmetical sequent. If there exists an expansion proof with shallow sequent Γ, then $\vdash^l_{\omega^2} \Gamma^\infty$ with l finite.*

3.3.2 Consistency of PA

From our previous results it is possible to prove :

Theorem 5. *PA is consistent.*

Proof. Indeed, assuming there is a proof of the empty sequent in PA :

- We know by Theorem 1 that there is an expansion proof with the empty shallow sequent. This is provable in a very weak sub-system of PA.

- From this we know by Lemma 6 that $\vdash^k_{\omega^2} \emptyset$ with k finite. This is provable in a very weak sub-system of PA.
- Using Theorem 4 and the fact that $\omega^2 < \varepsilon_0$, we know that $\vdash^0_{\omega_k(\omega^2)} \emptyset$. This step uses induction up to ε_0, indeed we need to perform nested inductions on k and on $\omega_k(\omega^2)$, the smallest possible bound is ε_0.
- It is easy to see that this is a contradiction : a strategy without cut for the empty sequent is empty, and therefore it can not be winning.

$\square$

3.3.3 The failure of parallelism

Recall that Lemma 4 (which states that if for all i we have $\vdash^\beta_{\alpha_i} \Gamma, A_i$ then $\vdash^\beta_{\sup^*_i \alpha_i} \Gamma, \bigwedge_i A_i$) is unparallelising the strategies.

So our proof is not fully satisfying as a proof of consistency of PA using expansion proofs, because it uses Lemma 4. This means that we are using implicitly the fact that an infinitary expansion proof can be sequentialised into a proof in LK_∞, allowing us to prove cut-elimination as for proofs in LK_∞. Because of this, our normalisation procedure is not confluent in a bad way. In fact almost every choices of moves done in order to perform inductions are significant for the output.

4 Conclusion

We have presented finitary expansion proofs with induction. We have proven them equivalent to PA. Then we have introduced infinitary expansion proofs called strategies and we have shown a cut-elimination theorem for them, from which we have deduced consistency of PA. What should we do next ?

Firstly a parallel cut-elimination procedure for strategies should be defined.

Then it would be nice to have a result proving consistency of PA using finitary expansion proofs only, for example by showing that the cut-elimination procedure of Definition 10 together with the rule eliminating inductions are terminating. This could possibly be done using strategies, either explicitly by using the theorems in this report, or implicitly as a supplementary insight.

On the other hand, strategies are interesting in their own right. They are in some ways more natural than finitary expansion proofs with inductions, partly because they have a cut-elimination procedure. They define a game-theoretic semantic for arithmetic equivalent to the usual semantic, which emphasises that the number of times Eve will change her mind can not be bounded a priori.

One interesting next step could be to use the techniques presented here to study (fragments of) second-order arithmetic. What kind of game-theoretic semantics for second order arithmetic would we get ?

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Here we present two appendices with the proofs of all the results from the main text. The definitions present in the main text are not recalled. The lemmas and theorems are recalled, and are numbered as in the main text : the numbering in the appendices is not linear.

Appendix A is about finitary expansion proofs, and Appendix B about the infinitary one.

A Properties of expansion proofs with induction

In Section A.1 we present some auxiliary constructions for expansion proofs. Then they are used in Section A.2 to prove soundness and completeness of expansion proofs with respect to PA. In Section A.3 we define a partial cut-elimination procedure. Moreover we define an elimination procedure for inductions on closed terms. We prove weak normalisation of the partial cut-elimination procedure, but we leave the normalisation of this procedure together with the elimination of inductions as an open problem.

The proofs in this appendix are very similar to the proofs in [1].

A.1 Auxiliary constructions

We present some technical tools useful to manipulate expansion proofs.

A.1.1 Substitutions

For $\mathcal{P}$ an expansion pre-proof and σ a substitution, we define $\mathcal{P}\sigma$ as usual.

Note that there is no guarantee that $\mathcal{P}\sigma$ will be an expansion pre-proof, even if $\mathcal{P}$ is. For example an eigenvariable could appear twice in $\mathcal{P}\sigma$, breaking regularity, or the σ could induce a cycle in the dependency relation of $\mathcal{P}\sigma$.

Lemma 7. *Let $\mathcal{P}$ be an expansion pre-proof.*

- $\mathrm{Dp}(\mathcal{P}\sigma) = \mathrm{Dp}(\mathcal{P})\sigma$
- $\mathrm{Sh}(\mathcal{P}\sigma) = \mathrm{Sh}(\mathcal{P})\sigma$

Proof. The first affirmation is a straightforward induction on $\mathcal{P}$.

The second is even easier, by case analysis. □

A.1.2 Expansion proofs with merge

Merges are used to perform contractions in an expansion proof. The main result is that it is possible to transform any expansion proof with merge into an expansion proof without. This is obtained by rewriting the expansion proof with merge, pushing merges upward.

All the definitions and proofs in this section can be applied to non-regular proofs, i.e. proofs having multiple occurrences of the same eigenvariable. This will be used in Section A.3.2.

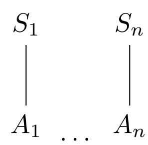
Definition 29. *We define expansion trees with merge the same as expansion trees (see Definition 1), with a new construction :*

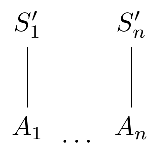
If E, E' are expansion trees with merge with $\mathrm{Sh}(E) = \mathrm{Sh}(E')$ then $E \sqcup E'$ is an expansion tree with merge with $\mathrm{Sh}(E \sqcup E') = \mathrm{Sh}(E)$.

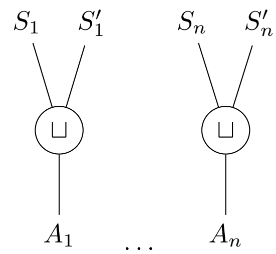
We add $\mathrm{Dp}(E \sqcup E') = \mathrm{Dp}(E) \vee \mathrm{Dp}(E')$.

We define expansion pre-proofs with merge and expansion proofs with merge exactly as expansion pre-proofs and expansion proofs.

So for example it is possible to build an expansion proof with shallow sequent $A_1, .., A_n$ out of two such proofs.

Assume that we have 

and 

Then we can build 

It is easy to check that this is an expansion proof with merge.

Now we want to be able to eliminate merge from an expansion proof without changing its shallow sequent. In order to do so we define a reduction on expansion proofs with merge.

Definition 30. *We define $\overset{\sqcup}{\longmapsto}$ as the context closure of the rules listed below.*

The regular merging rules :

- $L \sqcup L \overset{\sqcup}{\longmapsto} L$
- $(E_1 \circ E'_1) \sqcup (E_2 \circ E'_2) \overset{\sqcup}{\longmapsto} (E_1 \sqcup E'_1) \circ (E_2 \sqcup E'_2)$
- $(\exists x.A +^{t_1} E_1 + \ldots +^{t_n} E_n) \sqcup (\exists x.A +^{s_1} F_1 + \ldots +^{s_k} F_k)$
  $\overset{\sqcup}{\longmapsto} \exists x.A +^{t_1} E_1 + \ldots +^{t_n} E_n +^{s_1} F_1 + \ldots +^{s_k} F_k$
- $\exists x.A +^{t_1} E_1 +^{t_2} E_2 + \ldots +^{t_n} E_n \overset{\sqcup}{\longmapsto} \exists x.A +^{t_1} E_1 \sqcup E_2 + \ldots +^{t_n} E_n$
  if $t_1 = t_2$.

The substituting merging rule :

- $(\forall x.A +^{\alpha} E) \sqcup (\forall x.A +^{\beta} E') \overset{\sqcup}{\longmapsto} \forall x.A +^{\alpha} E \sqcup E'$
  and β is renamed everywhere in the context to α.

The global merging rules :

- $\mathcal{P}, (E_1, E_2), (E'_1, E'_2) \overset{\sqcup}{\longmapsto} \mathcal{P}, (E_1 \sqcup E'_1, E_2 \sqcup E'_2)$ *if* $\mathrm{Sh}(E_1) = \mathrm{Sh}(E'_1)$.
- $\mathcal{P}, (E_0, E_s, E), (E'_0, E'_s, E') \overset{\sqcup}{\longmapsto} \mathcal{P}, (E_0 \sqcup E'_0, E_s \sqcup E'_s, E \sqcup E')$
  if $\mathrm{Sh}(E) = \mathrm{Sh}(E')$*, meaning they are the same formula $A(t)$ with the same occurrences of t indicated.*

Lemma 8. *The relation $\overset{\sqcup}{\longmapsto}$ maps an expansion pre-proof with merge to an expansion pre-proof with merge with the same shallow sequent.*

Proof. This is a straightforward verification. □

Lemma 9. *The relation $\overset{\sqcup}{\longmapsto}$ is strongly normalising.*

Proof. We define the weight of a $\sqcup$ or $\exists$ node as the number of nodes above it, not counting $\sqcup$ nodes. Let us denote by $w_i(\mathcal{P})$ the number of nodes of weight i in $\mathcal{P}$. When a rule is applied :

- If we are dealing with a global merging rule the number cuts and inductions in the proof is decreasing.

- Otherwise the number of cuts and inductions is constant. If we are dealing with the substituting merging rule or the fourth regular merging rule, the number of expansions decreases.

- Otherwise the number of expansions is constant. For all the rules left, we have that $(w_i(\mathcal{P}))_{i\in\mathbb{N}}$ lexicographically ordered decreases.

$\square$

Lemma 10. *If $\mathcal{P}$ is in normal form for $\stackrel{\sqcup}{\longmapsto}$, then it has no merge node.*

Proof. If there is a merge, let $E_1 \sqcup E_2$ be an uppermost one. Since it is uppermost neither of the E_i begins with a merge. We denote $\mathrm{Sh}(E_1) = \mathrm{Sh}(E_2)$ by A. By case distinction on A, the merge can be reduced. $\square$

Lemma 11. *The relation $\stackrel{\sqcup}{\longmapsto}$ is confluent.*

Proof. We omit this the tedious verification that it is strongly confluent. $\square$

Lemma 12. *Let $\mathcal{P}[\_]$ be an expansion proof with a hole. If $Q^- \vdash \overline{\mathrm{Dp}(E)} \vee \mathrm{Dp}(E')$, then $Q^- \vdash \overline{\mathrm{Dp}(\mathcal{P}[E])} \vee \mathrm{Dp}(\mathcal{P}[E'])$.*

Proof. It is enough to check that for any context $\Gamma[\_]$ with a hole and formulas built using $\vee$, $\wedge$, atoms and their negations, we have that $Q^- \vdash \overline{A} \vee A'$ implies $Q^- \vdash \overline{\Gamma[A]} \vee \Gamma[A']$. This is easy. $\square$

Lemma 13. *If $\mathcal{P} \stackrel{\sqcup}{\longmapsto} \mathcal{P}'$ and $Q^- \vdash \mathrm{Dp}(\mathcal{P})$, then $Q^- \vdash \mathrm{Dp}(\mathcal{P}')$.*

Proof. For the regular merging and global merging rules, this is a straightforward consequences of Lemma 12.

For the substituting merging rule we have using Lemma 7 that $\mathrm{Dp}(\mathcal{P}') = \mathrm{Dp}(\mathcal{P})[\alpha/\beta]$, so we can conclude by noticing that $Q^- \vdash \Gamma$ implies $Q^- \vdash \Gamma\sigma$. $\square$

Lemma 14. *If $\mathcal{P} \stackrel{\sqcup}{\longmapsto} \mathcal{P}'$ and $\mathcal{P}$ is acyclic, so is $\mathcal{P}'$.*

Proof. The first three regular merging rules do not change the dependency relation. The other reductions identify two nodes with the same predecessors in the dependency relation. This cannot introduce a new cycle. $\square$

Lemma 15. *Let $\mathcal{P}$ be an expansion proof with merge. Then its unique normal form under $\stackrel{\sqcup}{\longmapsto}$ is an expansion proof with the same shallow sequent. It is denoted by $\mathcal{P}!$.*

Proof. This sums up the preceding lemmas. We use Lemma 13 to build the necessary proof of $Q^- \vdash \mathrm{Dp}(\mathcal{P}!)$ from the given proof of $Q^- \vdash \mathrm{Dp}(\mathcal{P})$. $\square$

We will use this lemma a lot in the following.

Assume given two expansion pre-proofs $\mathcal{P}, \mathcal{P}'$ with the same shallow sequent $A_1, \ldots, A_n$. Assume $\mathcal{P} = E_1, \ldots, E_n, C_1, \ldots, C_k$ and $\mathcal{P}' = E'_1, \ldots, E'_n, D_1, \ldots, D_l$ with $\mathrm{Sh}(E_i) = \mathrm{Sh}(E'_i) = A_i$ and C_i and D_i cuts or inductions. Then we denote $E_1 \sqcup E'_1, \ldots, E_n \sqcup E'_n, C_1, \ldots, C_k, D_1, \ldots, D_l$ by $\mathcal{P} \sqcup \mathcal{P}'$

A.1.3 Weakening

We present an auxiliary construction useful to perform weakening on expansion proofs.

Definition 31. *Let A be a formula, we define an expansion tree* $\mathrm{wk}(A)$ *with* $\mathrm{Sh}(\mathrm{wk}(A)) = A$ *by induction on A :*

- $\mathrm{wk}(s = t) = (s = t)$.
- $\mathrm{wk}(A \circ B) = \mathrm{wk}(A) \circ \mathrm{wk}(B)$ *where* $\circ \in \{\wedge, \vee\}$.
- $\mathrm{wk}(\forall x.A(x)) = \forall x.A(x) +^{\alpha} \mathrm{wk}(A(\alpha))$ *where α is a fresh variable.*
- $\mathrm{wk}(\exists x.A(x)) = \exists x.A(x)$*, that is an existential node without expansion.*

For a sequent $\Gamma = A_1, \ldots, A_n$*, we define* $\mathrm{wk}(\Gamma) = \mathrm{wk}(A_1), \ldots, \mathrm{wk}(A_n)$.

Lemma 16. *For any finite set of formulas Γ, we have that* $\mathrm{wk}(\Gamma)$ *is an expansion pre-proof with* $\mathrm{Sh}(\mathrm{wk}(\Gamma)) = \Gamma$.

Let $A_1, \ldots, A_n$ be the atoms in Γ. If $Q^- \vdash A_1, \ldots, A_n$, then $\mathrm{wk}(\Gamma)$ *is an expansion proof.*

Proof. This is immediate. □

A.2 Soundness and completeness with respect to PA

A.2.1 A definition of PA

Definition 32. *We give the definition of a one-sided sequent calculus for PA.*

$$\frac{}{\Gamma, A, \overline{A}}\ Ax$$

$$\frac{}{\Gamma, A_1, \ldots, A_n}\ Ax$$

where $A_1, \ldots, A_n$ is an instantiation of an axiom of Q^-.

$$\frac{\Gamma, A \qquad \Gamma, \overline{A}}{\Gamma}\ Cut$$

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \wedge$$

$$\frac{\Gamma, A, B}{\Gamma, A \vee B}\ \vee$$

$$\frac{\Gamma, A(\alpha)}{\Gamma, \forall x.A(x)}\ \forall$$

Where α is not appearing in the end-sequent.

$$\frac{\Gamma, \exists x.A(x), A(t)}{\Gamma, \exists x.A(x)}\ \exists$$

$$\frac{\Gamma, A(0) \qquad \Gamma, \forall x.\overline{A}(x) \vee A(s(x)) \qquad \Gamma, \overline{A}(t)}{\Gamma}\ \mathrm{Ind}$$

Where t is any term.

We write $\vdash_{PA} \Gamma$ to say that there exists a proof of Γ in this system.

A.2.2 Completness

This is a relatively straightforward translation. We just need to be a bit careful about variables because they are not handled the same way in sequent calculus and expansion proofs.

Lemma 17. *If* $\vdash_{PA} \Gamma$*, then there exists an expansion proof* $\mathcal{P}$ *with* $\mathrm{Sh}(\mathcal{P}) = \Gamma$.

Proof. We proceed by induction on the proof of Γ. We will only treat a few cases, the rest can be found in [1].

- For an axiom proving Γ, we know from Lemma 16 that $\mathrm{wk}(\Gamma)$ is an expansion proof.
- If we are dealing with a cut, we need to build an expansion proof with shallow sequent Γ from :

  - An expansion proof $\mathcal{P}, E_A$ with $\mathrm{Sh}(\mathcal{P}) = \Gamma$ and $\mathrm{Sh}(E_A) = A$.
  - An expansion proof $\mathcal{P}', E_{\overline{A}}$ with $\mathrm{Sh}(\mathcal{P}') = \Gamma$ and $\mathrm{Sh}(E_{\overline{A}}) = \overline{A}$.

  It is easy to check that $Q^- \vdash \mathrm{Dp}(\mathcal{P} \sqcup \mathcal{P}', (E_A, E_{\overline{A}}))$. To check that it is acyclic, notice that since there are no eigenvariable of $\mathcal{P}$ or $\mathcal{P}'$ in A, the new node introduced is minimal, so there are no new cycles.

  Then $(\mathcal{P} \sqcup \mathcal{P}', (E_A, E_{\overline{A}}))!$ is an expansion proof with shallow sequent Γ.

- For an induction, assume given :

  - An expansion proof $\mathcal{P}, E_0$ with $\mathrm{Sh}(\mathcal{P}) = \Gamma$ and $\mathrm{Sh}(E_0) = A(0)$.
  - An expansion proof $\mathcal{P}', E_s$ with $\mathrm{Sh}(\mathcal{P}') = \Gamma$ and $\mathrm{Sh}(E_s) = \forall x.\overline{A}(x) \vee A(s(x))$.
  - An expansion proof $\mathcal{P}'', E$ with $\mathrm{Sh}(\mathcal{P}'') = \Gamma$ and $\mathrm{Sh}(E) = \overline{A}(t)$.

  Then $\mathcal{P} \sqcup \mathcal{P}' \sqcup \mathcal{P}'', (E_0, E_s, E)$ is an expansion pre-proof with merge.

  We can check that $Q^- \vdash \mathrm{Dp}(\mathcal{P} \sqcup \mathcal{P}' \sqcup \mathcal{P}'', (E_0, E_s, E))$. Indeed this sequent is equivalent to $Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(\mathcal{P}'), \mathrm{Dp}(\mathcal{P}''), \mathrm{Dp}(E_0) \wedge \mathrm{Dp}(E_s) \wedge \mathrm{Dp}(E)$. We can conclude using the fact that $Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(E_0)$, that $Q^- \vdash \mathrm{Dp}(\mathcal{P}'), \mathrm{Dp}(E_s)$, and that $Q^- \vdash \mathrm{Dp}(\mathcal{P}''), \mathrm{Dp}(E)$.

  It is acyclic because no eigenvariable can appear in $A(t)$ so the new induction node is minimal and we do not introduce any cycle.

  So $(\mathcal{P} \sqcup \mathcal{P}' \sqcup \mathcal{P}'', (E_0, E_s, E))!$ is an expansion proof with shallow sequent Γ.

□

A.2.3 Soundness

Now we prove that it is possible to extract a sequent proof in PA from an expansion proof. Since the dependency relation on events is acyclic, it is possible to consider a minimal event in any expansion proof. Such an event should be translated first.

Definition 33. *We define an intermediate system* **LKE** *between PA and expansion proofs by the rules :*

$$\frac{}{A_1, \ldots, A_n}\, Ax$$

where the A_i *are atomic and* $Q^- \vdash A_1, \ldots, A_n$.

$$\frac{\mathcal{P}, E_1 \qquad \mathcal{P}, E_2}{\mathcal{P}, E_1 \wedge E_2}\, \wedge$$

$$\frac{\mathcal{P}, E_1, E_2}{\mathcal{P}, E_1 \vee E_2}\, \vee$$

$$\frac{\mathcal{P}, E_1 \qquad \mathcal{P}, E_2}{\mathcal{P}, (E_1, E_2)}\ Cut$$

if $\mathrm{Sh}(E_1) = \overline{\mathrm{Sh}(E_2)}$

$$\frac{\mathcal{P}, E}{\mathcal{P}, \forall x.A(x) +^\alpha E}\ \forall$$

if $\mathrm{Sh}(E) = A(\alpha)$*, where* α *does not occur in* $\mathrm{Sh}(\mathcal{P})$ *and* A*.*

$$\frac{\mathcal{P}, E_0 \qquad \mathcal{P}, E_s \qquad \mathcal{P}, E}{\mathcal{P}, (E_0, E_s, E)}\ Ind$$

if $\mathrm{Sh}(E_0) = A(0)$*,* $\mathrm{Sh}(E_s) = \forall x.\overline{A}(x) \vee A(s(x))$ *and* $\mathrm{Sh}(E) = \overline{A}(t)$*.*

$$\frac{\mathcal{P}, \exists x.A(x) +^{t_1} E_1 + \ldots +^{t_n} E_n, E_{n+1}}{\mathcal{P}, \exists x.A(x) +^{t_1} E_1 + \ldots +^{t_n} E_n +^{t_{n+1}} E_{n+1}}\ \exists$$

if $\mathrm{Sh}(E_{n+1}) = A(t_{n+1})$*.*

$$\frac{\mathcal{P}}{\mathcal{P}, \exists x.A(x)}\ \exists_0$$

Lemma 18. *If* $\vdash_{\mathbf{LKE}} \mathcal{P}$*, then* $\vdash_{PA} \mathrm{Sh}(\mathcal{P})$*.*

Proof. By induction. □

Lemma 19. *Let* $\mathcal{P}$ *be an expansion proof, then* $\vdash_{\mathbf{LKE}} \mathcal{P}$*.*

Proof. We proceed by induction on the number of nodes in $\mathcal{P}$.

- If $\mathcal{P}$ contains a $\exists x.A(x)$ without expansion, then we can use the $\exists_0$ rule.
- If $\mathcal{P}$ contains only atoms, then $Q^- \vdash \mathrm{Dp}(\mathcal{P})$ so we can use the Ax rule.
- If $\mathcal{P} = \mathcal{P}', E_1 \vee E_2$ then $\mathcal{P}', E_1, E_2$ is an expansion proof. By induction $\vdash_{\mathbf{LKE}} \mathcal{P}', E_1, E_2$ and we can conclude using the $\vee$ rule.
- If $\mathcal{P} = \mathcal{P}', E_1 \wedge E_2$ then $\mathcal{P}', E_1$ and $\mathcal{P}', E_2$ are expansion proofs, by induction hypothesis $\vdash_{\mathbf{LKE}} \mathcal{P}', E_i$ for $i \in \{1, 2\}$ and we can conclude using the $\wedge$ rule.

Otherwise any expansion tree in $\mathcal{P}$ begins with $\exists x.A(x) +^{t_1} E_1 + .. +^{t_n} E_n$, $\forall x.A(x) +^\alpha E$ or is an atom. Moreover there is at least one event in $\mathcal{P}$. We take a minimal event in $\mathcal{P}$, then either it is at the root of an expansion tree of the proof or it is a cut or an induction.

- If $\mathcal{P} = \mathcal{P}', (E_1, E_2)$ with (E_1, E_2) minimal, then there are no eigenvariable of $\mathcal{P}$ in $\mathrm{Sh}(E_1)$, otherwise $\forall\alpha <_{\mathcal{P}} (E_1, E_2)$ for some α, contradicting minimality. Therefore no eigenvariable of $\mathcal{P}$ appear in $\mathrm{Sh}(\mathcal{P}'), \mathrm{Sh}(E_i)$ for $i \in \{1, 2\}$. From this we see that $\mathcal{P}', E_i$ are expansion proofs, by induction $\vdash_{\mathbf{LKE}} \mathcal{P}', E_i$ so we can conclude using the cut rule.
- Similarly if $\mathcal{P} = \mathcal{P}', (E_0, E_s, E)$ with (E_0, E_s, E) minimal, then no eigenvariable of $\mathcal{P}$ appears in $\mathrm{Sh}(E)$ by minimality. We can conclude using induction hypothesis and the Ind rule.
- If $\mathcal{P} = \mathcal{P}', \forall x.A(x) +^\alpha E$ with $\forall\alpha$ minimal, then $\mathcal{P}', E$ is an expansion proof. We can conclude using $\forall$ rule, whose side condition is satisfied.
- If $\mathcal{P} = \mathcal{P}', \exists x.A(x) +^{t_1} +E_1 \ldots +^{t_n} E_n$ with for example $\exists t_1$ minimal, then by minimality no eigenvariable of $\mathcal{P}$ can occur in t_1. So $\mathcal{P}', E_1, \exists x.A(x) +^{t_2} E_2 + .. +^{t_n} E_n$ is an expansion proof and is provable in **LKE** by induction hypothesis. We can conclude using the $\exists$ rule.

□

Theorem 1. *There exists an expansion proof* $\mathcal{P}$ *with* $\mathrm{Sh}(\mathcal{P}) = \Gamma$ *if and only if* $\vdash_{PA} \Gamma$

Proof. This is a consequence of Lemmas 17, 18 and 19 □

A.3 Partial cut-elimination

Here we define a partial cut-elimination procedure for expansion proofs and show that it is correct and terminating. It is called partial because it does not eliminate inductions. In Section A.3.5 we introduce a rule eliminating inductions on closed terms and show that it is correct.

A.3.1 Elimination of propositional cuts

The rules to eliminate cuts on atoms and cuts of the form $(A \wedge B, \overline{A} \vee \overline{B})$ are very simple. They are called propositional cut-reductions.

Definition 34. *We define some rewriting rules called propositional cut-reductions.*

$$(L, \overline{L}), \mathcal{P} \mapsto \mathcal{P}$$

where L is an atomic formula.

$$(E_1 \wedge E_2, E_1' \vee E_2'), \mathcal{P} \mapsto (E_1, E_1'), (E_2, E_2'), \mathcal{P}$$

Lemma 20. *If $\mathcal{P} \mapsto \mathcal{Q}$ by a propositional cut-reduction and $\mathcal{P}$ is an expansion proof, then $\mathcal{Q}$ is an expansion pre-proof with* $\mathrm{Sh}(\mathcal{P}) = \mathrm{Sh}(\mathcal{Q})$.

Proof. Immediate. □

Lemma 21. *If $\mathcal{P} \mapsto \mathcal{Q}$ by a propositional cut-reduction and $\mathcal{P}$ is an expansion proof, then $\mathcal{Q}$ is an expansion proof with* $\mathrm{Sh}(\mathcal{P}) = \mathrm{Sh}(\mathcal{Q})$.

Proof. For the atomic reduction :

- We know that $Q^- \vdash L \wedge \overline{L}, \mathrm{Dp}(\mathcal{Q})$, so $Q^- \vdash \mathrm{Dp}(\mathcal{Q})$ using a cut.
- The events and their dependency relations are unchanged.

For the $\wedge/\vee$ reduction :

- By hypothesis $Q^- \vdash \mathcal{P}', \mathrm{Dp}(E_1) \wedge \mathrm{Dp}(E_2) \wedge (\mathrm{Dp}(E_1') \vee \mathrm{Dp}(E_2'))$, so using propositional calculus $Q^- \vdash \mathcal{P}', \mathrm{Dp}(E_1) \wedge \mathrm{Dp}(E_1'), \mathrm{Dp}(E_2) \wedge \mathrm{Dp}(E_2')$.
- The events of $\mathcal{P}$ are the events of the $\mathcal{Q}$ with two nodes identified : if there is a cycle in $\mathcal{Q}$ there is one in $\mathcal{P}$.

□

A.3.2 Weakly regular pre-proofs

We will use weakly regular pre-proofs as an auxiliary tool for cut-elimination of quantified formulas.

A weakly regular pre-proof is an expansion pre-proof with merge except that an eigenvariable can appear multiple times if those multiple occurrences are merged during the reduction by $\overset{\sqcup}{\longmapsto}$. See [1] for a precise definition.

Definition 35. *A weakly regular proof is a weakly regular pre-proof which has an acyclic dependency relation together with a proof of its deep sequent.*

Lemma 22. *The normalisation of a weakly regular pre-proof under $\overset{\sqcup}{\longmapsto}$ is an expansion pre-proof.*

The normalisation of a weakly regular proof under $\overset{\sqcup}{\longmapsto}$ is an expansion proof.

Proof. Adapt Section A.1.2, or see [1]. □

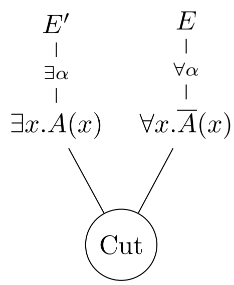


Figure 8: A pathological expansion tree

A.3.3 Elimination of quantified cuts

Definition 36. *We define the first-order cut-reductions.*

$$\mathcal{P}, (\exists x.A(x) +^{t_1} E_1 + \ldots +^{t_n} E_n, \forall x.\overline{A}(x) +^{\alpha} E)$$

$$\mapsto ((E_1, E\eta_1\sigma_1), \ldots, (E_n, E\eta_n\sigma_n), \mathcal{P} \sqcup \mathcal{P}\eta_1\sigma_1 \sqcup \ldots \sqcup \mathcal{P}\eta_n\sigma_n)!$$

where $\sigma_i(\alpha) = t_i$, and η_i are renaming eigenvariables γ such that $\forall\alpha < \forall\gamma$ to fresh variables.

It is not surprising that we need to duplicate things since existential nodes are implicit contractions. Note that there is one copy of $\mathcal{P}$ left untouched. This is necessary because we have no condition preventing objects like the expansion tree in Figure 8 to appear in an expansion proof. From a sequent point of view an existential formula is instantiated by an eigenvariable from another branch of the proof. Since it is not straightforward to characterise these phenomena we allow them, and we add this unnatural copy of $\mathcal{P}$ in the reduct.

Now we prove some properties of the reduct of an expansion proof.

Let us denote $(E_1, E\eta_1\sigma_1), \ldots, (E_n, E\eta_n\sigma_n), \mathcal{P} \sqcup \mathcal{P}\eta_1\sigma_1 \sqcup \ldots \sqcup \mathcal{P}\eta_n\sigma_n$ by $\mathcal{Q}$.

Lemma 23. *$\mathcal{Q}$ is a weakly regular pre-proof.*

Proof. First we check that $\mathrm{Sh}(E_i) = A(t_i)$ and $\mathrm{Sh}(E\eta_i\sigma_i) = \overline{A}(\alpha)\eta_i[\alpha/t_i]$ are dual. For this it is enough to check that any renamed γ does not appear in $\overline{A}(\alpha)$. Otherwise we would have $C < \forall\alpha < \forall\gamma < C$, that is a cycle. Note that the order of η_i and σ_i is important.

Next we need to check that it is legal to merge $\mathcal{P}$ and $\mathcal{P}\eta_i\sigma_i$ for any i. It is enough that neither α nor any renamed eigenvariable of $\mathcal{P}$ occur in $\mathrm{Sh}(\mathcal{P})$. This is true because no eigenvariable of the old proof occurs in $\mathrm{Sh}(\mathcal{P})$. For the same reason it is clear that no eigenvariable of $\mathcal{Q}$ can occur in the shallow sequent of $\mathcal{Q}$.

Now we need to check that two occurrences of the same eigenvariable γ not merged by $\stackrel{\sqcup}{\longmapsto}$ are renamed. If they are not merged then there is an event below them that was not merged. Consider the lowest one. We know that α or a renamed variable $> \forall\alpha$ occurs in this event otherwise it would have been merged. So the variables not merged come from a variable $> \forall\alpha$ and hence are renamed. ☐

Lemma 24. *The deep sequent of $\mathcal{Q}$ is valid.*

Proof. We know that

$$Q^- \vdash \mathrm{Dp}(\mathcal{P}), \left(\bigvee_{i=1}^{n} \mathrm{Dp}(E_i)\right) \wedge \mathrm{Dp}(E)$$

so $Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(E_1), \ldots, \mathrm{Dp}(E_n)$ and $Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(E)$. From this it is easy to build a proof of :

$$Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(\mathcal{P})\eta_1\sigma_1, \ldots, \mathrm{Dp}(\mathcal{P})\eta_n\sigma_n, \mathrm{Dp}(E_1) \wedge (\mathrm{Dp}(E)\eta_1\sigma_1), \ldots, \mathrm{Dp}(E_1) \wedge (\mathrm{Dp}(E)\eta_n\sigma_n)$$

☐

Lemma 25. *The dependency relation of $\mathcal{Q}$ is acyclic.*

Proof. Assume that there is a cycle in $\mathcal{Q}$, we will show that there is one in the old expansion proof. Indeed any event in the new trees can be assigned to an event in the old expansion proof in a natural way (all the $(E_i, E\eta_i\sigma_i)$ are assigned to the reduced cut). Assume that $v_1 <_{\mathcal{Q}} \ldots <_{\mathcal{Q}} v_n <_{\mathcal{Q}} v_1$, let $w_1, \ldots, w_n$ be their respective corresponding events in the old proof.

If γ is an eigenvariable and w is an event, we say that γ occurs in w if $w = \exists t$ and γ occurs in t, or w is a cut or induction with γ in its shallow formula.

- If $\forall\alpha \not<_{\mathcal{P}} w_i$ for all i, we show $w_i <_{\mathcal{P}} w_{i+1}$.

  If v_i dominates v_{i+1} this is clear.

  If $v_i = \forall\gamma <_{\mathcal{Q}} v_{i+1}$ and γ occur in v_{i+1}. Then γ is not renamed by η_i otherwise $\forall\alpha <_{\mathcal{P}} w_i$. We know that $\forall\gamma <_{\mathcal{Q}} w_{i+1}\eta_l\sigma_l$ for some l (or without substitution), so either γ appears in w_{i+1} and we are done, otherwise α occurs in w_{i+1} and σ_l rewrites α to a term where γ occurs. This is not possible since $\forall\alpha \not<_{\mathcal{P}} w_{i+1}$.

- Otherwise assume for example $\forall\alpha <_{\mathcal{P}} w_1$. We show that $\forall\alpha <_{\mathcal{P}} w_i$ for all i.

  If v_i dominates v_{i+1} this is easy.

  Otherwise $v_i = \forall\gamma <_{\mathcal{Q}} v_{i+1}$ and γ occurs in v_{i+1}. Then w_i is in the domain of η_l by induction. If $\gamma = \eta_l(\beta)$ is fresh, then β occurs in w_{i+1} (since $\eta_l(\beta)$ does not occur in any t_i) and we can conclude. If γ is not fresh, it may occur in t_i and maybe we have that α occurs in w_{i+1}, but we can conclude as well in this case.

- Now assume given a cycle where $\forall\alpha <_{\mathcal{P}} w_i$ for all i. We define the l-component of $\mathcal{Q}$ as the part where $\eta_l\sigma_l$ is applied.

  We show by induction that if v_1 is in the l-component for some l, so are all the v_j. Assume v_i in the l-component.

  If v_{i+1} is dominated by v_i it is clear that v_{i+1} is in the l-component.

  Otherwise $v_i = \forall\gamma <_{\mathcal{P}} v_{i+1}$ is a variable which has to be fresh (we have that $w_i >_{\mathcal{P}} \forall\alpha$ and v_i is in the l-component), and so v_{i+1} is in the same component.

- Finally assume given a cycle where $\forall\alpha <_{\mathcal{P}} w_i$ for all i, and which stays in the same component (possibly the component where no substitution was applied). We lift such a cycle inductively.

  If v_i dominates v_{i+1} it is clear.

  Otherwise $v_i = \forall\gamma < v_{i+1}$ and since they are in the same component it is not possible that γ occurs in v_{i+1} because of a σ_l, so we know that $w_i < w_{i+1}$.

□

Lemma 26. *The first-order cut-reduction of Definition 36 maps an expansion proof to an expansion proof with the same shallow sequent.*

Proof. This sums up the two previous sections. □

A.3.4 Termination

Now we show that the cut-reductions presented in the previous three sections form a weakly normalising rewriting system.

Lemma 27. *If $\mathcal{P} \mapsto \mathcal{P}'$ by reducing a quantified cut C, then the cuts C' duplicated in $\mathcal{P}'$ are such that $C <_{\mathcal{P}} C'$.*

Proof. Let C be the reduced cut. If a cut C' is such that $C \not<_{\mathcal{P}} C'$, then the shallow formula of C' is unaffected by the η_i and σ_i, otherwise it would have an eigenvariable γ occurring in its shallow formula such that $\gamma = \alpha$ or $\forall\gamma > \forall\alpha$, and in both cases $C < \forall\alpha \leq \forall\gamma < C'$ in $\mathcal{P}$. $\square$

Theorem 6. *The propositional cut-reductions together with the first order cut-reductions are weakly normalising.*

Proof. The depth of the formula of a cut C is called the depth of C. For an expansion proof $\mathcal{P}$ and a natural number i, we denote by $w_i(\mathcal{P})$ the number of cuts of depth i in $\mathcal{P}$. Let $\mathcal{P}$ be an expansion proof with some cuts, we want to show that there exists $\mathcal{P}'$ such that $\mathcal{P} \mapsto \mathcal{P}'$ and $(w_i(\mathcal{P}))_{i\in\omega} > (w_i(\mathcal{P}'))_{i\in\omega}$ for the lexicographic order. It is easy to infer weak normalisation from this.

- Assume there is a cut in $\mathcal{P}$ on an atom or on $A \wedge B$ for some A and B. In this case we reduce it, and the decreasing of $(w_i(\mathcal{P}))_{i\in\omega}$ is obvious.
- Otherwise all the cuts are on quantified formulas. In this case we choose a cut C of maximal depth, which is minimal with respect to the dependency relation among those of maximal depth. By Lemma 27 we know that the duplicated cuts are higher than C. Therefore the depths of the duplicated cuts are strictly smaller than the depth of C. Moreover the new cuts created by the reduction have smaller depth than C as well, therefore the number of cuts of maximal depth is strictly decreasing.

$\square$

Note that during this process the inductions are not eliminated and can be duplicated. Therefore this result is similar to the result stating that in PA, we can eliminate cuts which are not right before an induction, by using the usual cut-elimination procedure for pure first-order logic. This is presented in the context of sequent calculus in the first chapter of [5] under the name of *free-cut elimination*.

A.3.5 What about induction ?

Now we define a reduction for inductions. We show its correctness, but do not prove any termination result.

Definition 37. *A closed induction is an induction (E_0, E_s, E) with $\mathrm{Sh}(E) = A(t)$ where t is closed.*

Lemma 28. *If t is a closed term, there exists a unique $n \in \omega$ such that $Q^- \vdash s^n(0) = t$*

Proof. By induction on t. $\square$

Lemma 29. *For all terms s, t and any quantifier-free formula $A(x)$, if $Q^- \vdash s = t$ then $Q^- \vdash \overline{A}(s), A(t)$*

Proof. It is enough to show $Q^- \vdash s \neq t, \overline{A}(s), A(t)$.

By induction on A :

- $A(x) = A_1(x) \wedge A_2(x)$.

$$\dfrac{\dfrac{\dfrac{Q^- \vdash s \neq t, \overline{A}_1(s), A_1(t)}{Q^- \vdash s \neq t, \overline{A}_1(s), \overline{A}_2(s), A_1(t)} \qquad \dfrac{Q^- \vdash s \neq t, \overline{A}_2(s), A_2(t)}{Q^- \vdash s \neq t, \overline{A}_1(s), \overline{A}_2(s), A_2(t)}}{Q^- \vdash s \neq t, \overline{A}_1(s), \overline{A}_2(s), A_1(t) \wedge A_2(t)}}{Q^- \vdash s \neq t, \overline{A}_1(s) \vee \overline{A}_2(s), A_1(t) \wedge A_2(t)}$$

- $A(x) = A_1(x) \vee A_2(x)$ is similar.

- For the atomic case, we assume $u(x), u'(x)$ two terms and show $Q^- \vdash s \neq t, u(s) \neq u'(s), u(t) = u'(t)$. It is enough to show by induction on $u(x)$ that $Q^- \vdash s \neq t, u(s) = u(t)$. This is straightforward.

$\square$

Lemma 30. *If $\mathcal{P}, (E_0, E_s, E)$ is an expansion proof, (E_0, E_s, E) a closed induction with $\mathrm{Sh}(E) = \overline{A}(t)$ and t closed, then there exists $n \in \omega$ such that $\mathcal{P}, (E_0, E_s, E[t/s^n(0)])$ is an expansion proof.*

Proof. By Lemma 28 there exists n such that $Q^- \vdash s^n(0) = t$. By Lemma 29 we know that $Q^- \vdash \mathrm{Dp}(\mathcal{P}, (E_0, E_s, E[t/s^n(0)]))$. Moreover the dependency relation is unchanged. $\square$

Using this lemma we can assume that all closed inductions are on a term of the form $s^n(0)$. Now we can reduce them.

Definition 38. *We define the closed induction reductions. Assume $\mathrm{Sh}(E) = A(s^n(0))$:*

$$\mathcal{P}, (E_0, \forall x.\overline{A}(x) \vee A(s(x)) +^\alpha E_1 \vee E_2, E)$$

$$\mapsto ((E_0, E_1\eta_0\sigma_0), (E_2\eta_0\sigma_0, E_1\eta_1\sigma_1), \ldots, (E_2\eta_{n-1}\sigma_{n-1}, E), \mathcal{P} \sqcup \mathcal{P}\eta_0\sigma_0 \sqcup \ldots \sqcup \mathcal{P}\eta_{n-1}\sigma_{n-1})!$$

where $\sigma_i(\alpha) = s^i(0)$ and η_i are renaming eigenvariables γ such that $\forall\alpha < \forall\gamma$ to fresh variables.

The proof of correcness is very similar to the case of quantified formulas. Let us denote $(E_0, E_1\eta_0\sigma_0), (E_2\eta_0\sigma_0, E_1\eta_1\sigma_1), \ldots, (E_2\eta_{n-1}\sigma_{n-1}, E), \mathcal{P} \sqcup \mathcal{P}\eta_0\sigma_0 \sqcup \ldots \sqcup \mathcal{P}\eta_{n-1}\sigma_{n-1}$ by $\mathcal{Q}$

Lemma 31. *$\mathcal{Q}$ is a weakly regular pre-proof.*

Proof. We need to check that $\mathrm{Sh}(E_2)\eta_i\sigma_i = \overline{\mathrm{Sh}(E_1)}\eta_{i+1}\sigma_{i+1}$. No variable in the domain of η_i can appear in $A(s^n(0))$ (i.e. in $\mathrm{Sh}(E)$), otherwise we would have $C < \forall\alpha < \forall\gamma < C$. So $\mathrm{Sh}(E_2)\eta_i\sigma_i = A(s(\alpha))\eta_i\sigma_i = A(s^{i+1}(0))$ and $\overline{\mathrm{Sh}(E_1)}\eta_{i+1}\sigma_{i+1} = \overline{A}(\alpha)\eta_{i+1}\sigma_{i+1} = \overline{A}(s^{i+1}(0))$. This reasoning works for the two other new cuts.

The rest of the proof is word for word the proof of Lemma 23.

$\square$

Lemma 32. *The deep sequent of $\mathcal{Q}$ is valid.*

Proof. We know that

$$Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(E_0) \wedge (\mathrm{Dp}(E_1) \vee \mathrm{Dp}(E_2)) \wedge \mathrm{Dp}(E)$$

From this we have that $Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(E)$, that $Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(E_1), \mathrm{Dp}(E_2)$ and that $Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(E_0)$. We need to show

$$Q^- \vdash \mathrm{Dp}(\mathcal{P}), \mathrm{Dp}(\mathcal{P})\eta_0\sigma_0, \ldots, \mathrm{Dp}(\mathcal{P})\eta_{n-1}\sigma_{n-1},$$

$$\mathrm{Dp}(E_0) \wedge \mathrm{Dp}(E_1)\eta_0\sigma_0, \mathrm{Dp}(E_2)\eta_0\sigma_0 \wedge \mathrm{Dp}(E_1)\eta_1\sigma_1, \ldots, \mathrm{Dp}(E_2)\eta_{n-1}\sigma_{n-1} \wedge \mathrm{Dp}(E)$$

We look at every possibilities for the $\wedge$. Either there are $\mathrm{Dp}(E_1)\eta_i\sigma_i, \mathrm{Dp}(E_2)\eta_i\sigma_i$ for some i and we can conclude (using $Q^- \vdash \mathcal{P}, \mathrm{Dp}(E_1), \mathrm{Dp}(E_2)$ and substitutions). Otherwise we have $\mathrm{Dp}(E_0)$ or $\mathrm{Dp}(E)$ and we can conclude as well. $\square$

Lemma 33. *The dependency relation of $\mathcal{Q}$ is acyclic.*

Proof. Assume we have a cycle $v_1 <_{\mathcal{Q}} v_2 <_{\mathcal{Q}} \ldots . <_{\mathcal{Q}} v_m <_{\mathcal{Q}} v_1$. Every event v_i has a natural corresponding event w_i in the old proof, with the new cuts attributed to the old induction. We want to show that $w_1 <_{\mathcal{P}} \ldots <_{\mathcal{P}} w_1$.

We show that $w_i <_{\mathcal{P}} w_{i+1}$.

- If $v_i <_{\mathcal{Q}} v_{i+1}$ because v_i dominates v_{i+1} it is straightforward.

- Otherwise $v_i = \forall\gamma <_{\mathcal{Q}} v_{i+1}$ because γ occurs in v_{i+1}. Then $\forall\gamma = w_i\eta_l\sigma_l <_{\mathcal{Q}} w_{i+1}\eta_{l'}\sigma_{l'}$ (possibly without some η). Either γ is not in the codomain of η_i and we are done, otherwise it is a fresh variable and we can conclude.

□

Lemma 34. *The closed induction reduction maps an expansion proof to an expansion proof with the same shallow sequent.*

Proof. This sums up this section. □

A natural question to ask is whether the rewriting system consisting of the propositional reductions, the first order reduction and closed induction reductions all together is normalising. This problem is difficult because it entails consistency of PA, and therefore must use induction up to ε_0.

Gentzen solved a similar problem for sequent calculus by assigning to any proof an ordinal $< \varepsilon_0$, in such a way that this ordinal decrease during the cut-elimination procedure. It is not possible to straightforwardly adapt Gentzen's method to expansion proofs because they are in some sense a parallelisation of sequent calculus proofs, and the ordinals assigned by Gentzen's method depend on the chosen sequentialisation of an expansion proof.

In order to prove a consistency result for PA using expansion proofs, we decided to consider infinitary expansion proofs.

B Properties of infinitary expansion proofs

In this appendix we present proofs of the properties of strategies.

B.1 Elementary properties

First of all we present basic properties.

B.1.1 The reduct of a strategy is a strategy

In Definition 21 we presented a reduction on strategies corresponding to moves played according to the strategy. We prove here that this reduction maps winning strategies to winning strategies.

Lemma 35. *If S is a strategy and $S \mapsto S'$ then S' is a strategy.*

Proof. If the reduction is of the form $S, \exists_{i\in I} k_i.S_i \mapsto S, \exists_{i\in I-\{l\}} k_i.S_i, S_l$ then it is obvious that the reduct obeys the finite reaction principle : only one event labelled $\emptyset$ was suppressed.

If the reduction is of the form $S, \forall_{i\in\omega} i.S_i \mapsto -_{e_l}(S, S_l)$ then the events labelled A in the reduct were labelled A or $A \cup \{e_l\}$ in $S, \forall_{i\in\omega} i.S_i$: there is only a finite number of them.

In both cases an infinite increasing sequence in the reduct can be translated to an infinite increasing sequence in the old strategy. □

Lemma 36. *If S is a winning strategy and $S \mapsto S'$ then S' is a winning strategy.*

Proof. This is an easy consequence of the definition of a winning strategy : a strategy is winning if and only if all its normal forms under $\mapsto$ contains $\top$. □

B.1.2 Plays are finite

We prove here that the reduction $\mapsto$ is strongly normalising. This will be the crucial point to show that a strategy can be sequentialised into a proof in LK_∞ : it shows that this unfolding process terminates. Since it was obvious that the sequentialisation terminates in the finitary case, Lemma 1 has no analogue in the finitary case.

Lemma 1. *The reduction $\mapsto$ is strongly normalising.*

Proof. From an infinite reduction $T_0 \mapsto T_1 \mapsto \cdots$, we will extract an infinite $<_{T_0}$-increasing sequence of nodes in T_0.

The edges of any reduct can be associated injectively to the edges of the old strategy in an obvious fashion :

- If $S, \forall_{i\in\omega} i.S_i \mapsto -_{e_l}(S, S_l)$ then any edge in S or S_l is associated to the corresponding edge in S or $\forall_{i\in\omega} i.S_i$. In this case we say that the edge e_l labelled by l was played.
- If $S, \exists_{i\in I} k_i.S_i \mapsto S, \exists_{i\in I-\{l\}} k_i.S_i, S_l$, then any edge in S, $\exists_{i\in I-\{l\}} k_i.S_i$ or S_l is associated to the corresponding edge in S or $\exists_{i\in I} k_i.S_i$. In this case we say that the edge labelled by k_l was played.

The played edges of all the reductions $T_i \mapsto T_{i+1}$ can be seen as edges in T_0 by composing these injections. The set of played edges in T_0 is infinite since an edge is played at every reduction $T_i \mapsto T_{i+1}$. This infinite set is called F.

Note that if an existential edge s is played, then all the universal edges in $\mathrm{Dep}(s)$ have been played before.

We order F by $<_F$ the smallest relation such that :

- An existential edge is smaller than its universal successors for $<_{T_0}$.
- A universal edge s is smaller than its universal successors for $<_{T_0}$. Moreover it is smaller than existential edges t in T_0 s.t. $s \in \mathrm{Dep}(t)$ and s was the last played universal edge in $\mathrm{Dep}(t)$. It is smaller than universal edges at the bottom of a cut-structure C s.t. $s \in \mathrm{Dep}(C)$ and s was the last played universal edge in $\mathrm{Dep}(C)$.

We now show that $<_F$ is a subrelation of $<_{T_0}$, and that F and $<_F$ form a finite family of finitely branching tree. Since F is infinite, this will be enough to conclude there is an infinite increasing sequence for $<_{T_0}$.

It is clear that $s <_F s'$ implies $s <_{T_0} s'$.

We check that any edge in F has at most one predecessor for $<_F$.

- If it is a universal edge s, then if it has a predecessor for $<_{T_0}$, it is its only predecessor for $<_F$. Otherwise it has no predecessor for $<_{T_0}$, and it is at the bottom of a structure or a cut-structure. Its only possible predecessor for $<_F$ is the last played universal edge in the label of its cut-structure.
- If it is an existential node s it can only have the last played universal edge in $\mathrm{Dep}(s)$ as a predecessor for $<_F$.

We show that any edge in F has a finite number of successors for $<_F$.

- If the edge is existential, it can only have its universal successors for $<_{T_0}$ as successors for $<_F$. But at most one of them is played.
- If the edge s is universal, then it can only have as successors :
    - Its universal successors for $<_{T_0}$ (there is at most one played).

- The existential edges s' such that s is the last played edge in $\mathrm{Dep}(s')$. There is a finite number of those because they are labelled by $\emptyset$ right after s is played, and there can only be a finite number of such edges by the finite reaction principle.

- The universal edges s' at the bottom of a cut-structure C such that that $s \in \mathrm{Dep}(C)$ and s was the last played universal edge in $\mathrm{Dep}(C)$. This is similar to the previous case.

We show that there is a finite number of edges without predecessors for $<_F$. Those edges can be :

- Existential edges labelled by $\emptyset$ in T_0. By the finite reaction principle there is a finite number of those.
- Universal edges at the bottom of a cut-structure labelled by $\emptyset$. By the finite reaction principle there is a finite number of such cut-structures, and therefore a finite number of such edges played.

□

In Definition 22 we define plays as sequences of reduction. Henceforth this lemma is stating that all plays are finite.

B.1.3 Composition of strategies

Now we present some lemmas proving that strategies can be composed in various ways. Recall that the ordinal of a strategy is the height of its tree of reduction under $\mapsto$. Moreover recall that $\vdash^\beta_\alpha \Gamma$ means that there is a winning strategy for Γ of ordinal $\leq \alpha$ with cuts on formulas of depth $< \beta$.

First an auxiliary result :

Lemma 37. *Assume S is a winning strategy for $\Gamma, \bigvee_n A_n$ with ordinal α, then there is a winning strategy for $\Gamma, \bigvee_n A_n$ with ordinal α containing a structure beginning by an $\exists$ node for $\bigvee_n A_n$.*

Proof. We associate to a structure S for $\bigvee_n A_n$ a set called $\Phi(S)$ of triples consisting of :

- a natural number l.
- a finite set of universal edges L.
- a structure for A_l.

We define $\Phi(S)$ inductively :

- $\Phi(\exists_{i\in I} k_i.S_i) = \{(k_i, \mathrm{Dep}(k_i), S_i) \mid i \in I\}$
- $\Phi(\sqcup_{i\in\omega} i.S_i) = \cup_{i\in\omega}\{(l, L \cup \{e_i\}, S_l) \mid (l, L, S_l) \in \Phi(S_i)\}$ where e_i is the universal edge corresponding to the $\sqcup$ edge labelled i in $\sqcup_{i\in\omega} i.S_i$.

The set $\Phi(S)$ is countable.

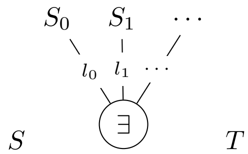
Then the structure S for $\bigvee_n A_n$ in the given strategy is associated to $\Phi(S)$ which can be renumbered $\{(l_i, L_i, S_i) \mid i \in I\}$ for some $I \subset \omega$. We claim that if we replace S by $\exists_{i\in I} l_i.S_i$ with L_i as the dependency of the edge labelled l_i, we obtain a winning strategy for $\Gamma, \bigvee_n A_n$, with the same ordinal as the given one.

It is easy to see that it obeys the finite reaction principle, and that it has no infinite increasing sequence.

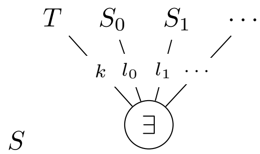
To see it is winning with the same ordinal, it is enough to observe that the two strategies behave precisely the same : a structure for A_k in S in the given strategy can be played if and only if the corresponding structure in the new strategy can be played. □

Lemma 3. *If $\vdash^\beta_\alpha \Gamma, \bigvee_n A_n, A_k$ then $\vdash^\beta_{\alpha+1} \Gamma, \bigvee_n A_n$.*

Proof. Assume we are given a winning strategy for $\Gamma, \bigvee_n A_n, A_k$. Using Lemma 37 we assume that this strategy is of the form :



Where S is a strategy for Γ, T is a structure for A_k and the structure beginning by $\exists$ is a structure for $\bigvee_n A_n$. We claim that the following strategy is a winning strategy for $\Gamma, \bigvee_n A_n$ with ordinal $\leq \alpha + 1$.



Here the dependency of the edge labelled by k is defined to be $\emptyset$.

It is easy to see that it is a strategy, and that it has no cuts on formulas of depth greater than β. We show by induction on α that it is a winning strategy with ordinal less than $\alpha + 1$.

- If $\alpha = 0$, then the new strategy has only one reduct which is in normal form and winning.
- Otherwise we show that the new strategy has reducts of ordinals $\leq \alpha$. One of this reduct is the old strategy which is winning and of ordinal α. The other ones are reducts of the old strategy (therefore of ordinals $< \alpha$) where the same transformation has been performed. Inductively they are of ordinal $\leq \alpha$ and winning.

$\square$

Lemma 4. *If for all natural numbers i we have $\vdash^{\beta}_{\alpha_i} \Gamma, A_i$, then $\vdash^{\beta}_{\sup^*_i \alpha_i} \Gamma, \bigwedge_k A_k$.*

Proof. Assume $\Gamma = B_1, \ldots, B_m$. Denote the given strategy for Γ, A_i by $T^i_1, \ldots, T^i_m, S_i, C^i_0, C^i_1, \ldots$ where :

- Each C^i_l is a cut. Note we assumed there was a countable number of them for notational convenience. There could be a finite number of them, this does not change anything in the proof.
- S_i a structure for A_i.
- T^i_l is a structure for B_l.

For S a structure or cut-structure and e a universal edge, let us denote $+_e(S)$ the operation of adding e to all the dependency of events in S (of course if S is a cut-structure this includes S which is itself an event in S).

We now define a new strategy. It contains $\forall_{i\in\omega} i. +_{e_i} (S_i)$ where e_i is the edge labelled by i.

For each natural number i we add to our strategy :

$+_{e_i}(C^i_0)$,$+_{e_i}(C^i_1)$,...

Moreover for each l natural number smaller than m, we add :

$\sqcup_{i\in\omega} i. +_{e_i} (T^i_l)$

where the $\sqcup$ node is labelled by the $\forall$ node in $\forall_{i\in\omega} i. +_{e_i} (S_i)$.

We claim this is a winning strategy for $\Gamma, \bigwedge_k A_k$ with the right ordinal. We call the set of events and universal edges of the new strategy coming from the strategy for Γ, A_i the i-component of the new strategy. First we observe that any event or universal edge in the new strategy is in one of the i-component, except the new edges e_i in $\forall_{i\in\omega} i. +_i (S_i)$. Moreover an event is in the i-component if and only if e_i appears in its dependency. We have built a winning strategy :

- It obeys the finite reaction principle. Indeed if there is an infinite number of event labelled the same, then there are infinitely many containing e_i for some i and therefore there was an infinite number of event with the same dependency in the given strategy for Γ, A_i.
- It is easy to check that if $s < t$ in the new strategy and s is in the i-component, so is t. Therefore an infinite increasing sequence in the new proof gives us an infinite increasing sequence in one of the i-component (because it has to meet one of the component at some point). Then we have an infinite increasing sequence in one of the given strategy.
- We show that it is winning and that it has the claimed ordinal. Indeed its only possible reductions are on the structure $\forall_{i\in\omega} i. +_i (S_i)$, and after this move it behaves precisely as the strategy for Γ, A_i for some i.

$\square$

Lemma 5. *Assume* $\vdash^{\beta}_{\alpha} \Gamma, A$ *and* $\vdash^{\beta}_{\alpha'} \Gamma, \overline{A}$*, let us denote* δ *the depth of* A*. Then* $\vdash^{\sup(\beta,\delta+1)}_{\sup(\alpha,\alpha')+1} \Gamma$.

Proof. This is a consequence of Lemma 4. $\square$

B.2 Soundness and completness with respect to LK$_\infty$

In this section we define the sequent calculus for infinitary formulas LK$_\infty$, and prove that it is equivalent to our strategies.

B.2.1 Definition of LK$_\infty$

The calculus LK$_\infty$ manipulates infinitary formulas.

Definition 39. *A proof of* Γ *in the sequent calculus* LK$_\infty$ *is a well-founded tree where the root is labelled by* Γ *and which is built using using the rules :*

$$\frac{}{\Gamma, \top}\ Ax$$

$$\frac{\Gamma, A \qquad \Gamma, \overline{A}}{\Gamma}\ Cut$$

$$\frac{\Gamma, A_0 \qquad \Gamma, A_1 \qquad \cdots}{\Gamma, \bigwedge_i A_i}\ \bigwedge$$

$$\frac{\Gamma, \bigvee_i A_i, A_k}{\Gamma, \bigvee_i A_i}\ \bigvee$$

B.2.2 Soundness

Lemma 38. *If there is a winning strategy for* Γ*, then* Γ *is provable in* LK$_\infty$.

Proof. By well-foundedness of the ordinals of strategies.

- If the proof is in normal form then Γ contains $\top$ and an axiom rule is okay to conclude.
- Otherwise there is a possible reduction :

  - $\Gamma = \Gamma', \bigwedge_i A_i$, and the structure for $\bigwedge_i A_i$ can be reduced. Then by induction hypothesis we have proofs in LK$_\infty$ of Γ, A_i for every i. We can conclude using the $\bigwedge$ rule.

  - The reduction is on a cut. By induction hypothesis we have proof in LK$_\infty$ of Γ, A and $\Gamma, \overline{A}$. We can conclude using a cut.

  - $\Gamma = \Gamma', \bigvee_i A_i$ and the structure for $\bigvee_i A_i$ can be reduced. Then for some k we have a proof in LK$_\infty$ of $\Gamma, \bigvee_i A_i, A_k$ by induction hypothesis. We conclude using the $\bigvee$ rule.

$\square$

B.2.3 Completeness

Here we use the lemmas of Section B.1.3.

Lemma 39. *If* Γ *is provable in* LK_∞*, there exists a winning strategy for* Γ*.*

Proof. By well-foundedness of the proofs in LK_∞.

- An axiom is easy using the structure $\top$ for $\top$ and $\bot$ for the other formulas (recall that $\bot$ is a structure for any formula).
- The case of $\bigvee$ is Lemma 3.
- The case of $\bigwedge$ is Lemma 4.
- The case of Cut is Lemma 5.

$\square$

Theorem 2. Γ *is provable in* LK_∞ *if and only if there exists a winning strategy for* Γ*.*

Proof. This is a direct consequence of Lemma 38 and Lemma 39. $\square$

B.3 Consistency of PA

In this section we give precise proofs of the auxiliary results needed to establish consistency of PA in Section 3.3.2.

B.3.1 Cut-elimination for consistency of PA

Here we prove a cut-elimination result for strategies. We closely follow the proof of cut-elimination for LK_∞ given in [20].

We use only elementary means and induction up to ε_0, so we do not prove the results in their most general form. We did not discuss the details of ordinal notations, i.e. how to present induction up to ε_0 in a syntactic and elementary way, which does not involve set theory. Such an elementary presentation of the whole Veblen hierarchy (Definition 41) can be found in [19].

We denote the depth of A by $d(A)$ (cf. Definition 27 in Section 3.2.1).

Definition 40. *We say that* Γ' *is deducible from* Γ *if one of the two following condition is true.*

- *Either* $\Gamma' = \Gamma, A$.
- *Otherwise* $\Gamma = \Gamma'', \bigwedge_i A_i$ *and* $\Gamma' = \Gamma'', A_k$.

Note that if S is a strategy for Γ and $S \mapsto S'$, then S' is a strategy for a sequent Γ' with Γ' deducible from Γ

Lemma 40. *If* Γ' *is deducible from* Γ *and* $\vdash^\beta_\alpha \Gamma, \Delta$*, then* $\vdash^\beta_\alpha \Gamma', \Delta$

Proof. We proceed by case distinction on the definition of deducible.

In the first case we do a weakening using the fact that $\bot$ is a structure for any formula.

In the second case we replace the structure S for $\bigwedge_i A_i$ (with A_k in Γ') in the strategy for Γ, Δ by an inductively defined structure $\Psi(S)$:

- $\Psi(\forall_{i\in\omega} i.S_i) = S_k$
- $\Psi(\sqcup_{l\in\omega} l.S_l) = \sqcup_{l\in\omega} l.\Psi(S_l)$ with the $\sqcup$ node corresponding to the same $\forall$ node as before.

It is easy to see that this is a structure for A_k, and that this defines a strategy. To prove that it is winning and of ordinal $\leq \alpha$, we perform a straightforward induction. $\square$

Let α and α' be two ordinals. Recall that their natural sum $\alpha\#\alpha'$ is defined using their Cantor normal forms :

- Assume $\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ with $\alpha_1 \geq \ldots \geq \alpha_n$.
- Assume $\alpha' = \omega^{\alpha'_1} + \ldots + \omega^{\alpha'_n}$ with $\alpha'_1 \geq \ldots \geq \alpha'_m$.

Let us denote $\delta_1 \geq \ldots \geq \delta_{n+m}$ the ordinals α_i for $1 \leq i \leq n$ and α'_j for $1 \leq j \leq m$ counted with multiplicities and sorted in decreasing order. Then we define $\alpha\#\alpha' = \omega^{\delta_1} + \ldots + \omega^{\delta_{n+m}}$.

Now we give the main lemma.

Lemma 41. *Assume that for A such that $d(A) \leq k$, we have $\vdash^k_\alpha \Gamma, A$ and $\vdash^k_{\alpha'} \Gamma, \overline{A}$, with $\alpha, \alpha' < \varepsilon_0$. Then $\vdash^k_{\alpha\#\alpha'} \Gamma$.*

Proof. We proceed by induction on $\alpha\#\alpha'$.

- If both are in normal form, Γ, A and $\Gamma, \overline{A}$ contains $\top$ so Γ contains $\top$, so $\vdash^0_0 \Gamma$.
- If $\vdash^k_\alpha \Gamma, A$ with a reduction not on A, than we have a family of $\vdash^k_{\alpha_i} \Gamma_i, A$ with Γ_i deducible from Γ and $\alpha_i < \alpha$. Then by Lemma 40 we know that $\vdash^k_{\alpha'} \Gamma_i, \overline{A}$ for all i, so that by induction hypothesis $\vdash^k_{\alpha_i\#\alpha'} \Gamma_i$. From this we can conclude $\vdash^k_{\sup^*_i \alpha_i\#\alpha'} \Gamma$. But $\sup^*_i \alpha_i\#\alpha' \leq \alpha\#\alpha'$ so we can conclude.
- It is similar to the previous case if $\vdash^k_{\alpha'} \Gamma, \overline{A}$ with a reduction not on $\overline{A}$
- Otherwise if both have a reduction in A and $\overline{A}$, we denote $A = \bigwedge_i A_i$ and $\overline{A} = \bigvee_i \overline{A_i}$. Then for all i we have $\vdash^k_{\alpha_i} \Gamma, A_i$ and for some l we have $\vdash^\beta_{\alpha''} \Gamma, \bigvee_i \overline{A_i}, \overline{A_l}$ with $\alpha_i < \alpha$ and $\alpha'' < \alpha'$. By induction hypothesis (and using a weakening on $\overline{A_l}$), we have that $\vdash^k_{\alpha\#\alpha''} \Gamma, \overline{A_l}$. By using a cut, since $d(\overline{A_l}) < k$ we have $\vdash^k_{\sup^*(\alpha\#\alpha'', \alpha_k)} \Gamma$. It is easy to conclude from this.
- The only case left is when one has a reduction on A, but the other is in normal form. In this case either Γ contains $\top$ and it is easy to conclude, or $A = \top$ contradicting the existence of a reduction on A or $\overline{A}$.

□

The main point of this lemma is that we only ask for $d(A) \leq k$ and not $d(A) < k$. A suitable iteration of this result will eliminate cuts. The restriction to $\alpha, \alpha' < \varepsilon_0$ is artificial, but this restricted form can be proven in PA using induction up to ε_0.

Lemma 42. *If $\vdash^{k+1}_\alpha \Gamma$ with $\alpha < \varepsilon_0$ then $\vdash^k_{\omega^\alpha} \Gamma$.*

Proof. We proceed by induction on α.

- If $\alpha = 0$, then the strategy is in normal form so $\vdash^0_0 \Gamma$.
- The strategy has a non-cut reduction.

  This reduction is on $\exists$, then $\Gamma = \Gamma', \bigvee_i A_i$ and $\vdash^{k+1}_{\alpha'} \Gamma', \bigvee_i A_i, A_l$ for $\alpha' < \alpha$. By induction hypothesis $\vdash^k_{\omega^{\alpha'}}, \Gamma', \bigvee_i A_i, A_l$. But $\omega^{\alpha'} < \omega^\alpha$ so we can conclude by Lemma 3.

  This reduction is on $\forall$, then $\Gamma = \Gamma', \bigwedge_i A_i$ and $\vdash^{k+1}_{\alpha_i} \Gamma', A_i$ for all i, with $\alpha_i < \alpha$. By induction hypothesis $\vdash^k_{\omega^{\alpha_i}} \Gamma', A_i$. Then by Lemma 4 we have $\vdash^k_{\sup^*_i \omega^{\alpha_i}} \Gamma$. But $\sup^*_i \omega^{\alpha_i} \leq \omega^\alpha$ so we can conclude.
- Otherwise The strategy has a cut-reduction. This means we have $\vdash^{k+1}_{\alpha_1} \Gamma, A$ and $\vdash^{k+1}_{\alpha_2} \Gamma, \overline{A}$ for some A with $d(A) \leq k$ and $\alpha_1, \alpha_2 < \alpha$. By induction hypothesis $\vdash^\beta_{\omega^{\alpha_1}} \Gamma, A$ and $\vdash^\beta_{\omega^{\alpha_2}} \Gamma, \overline{A}$.

  Using Lemma 41 we have $\vdash^k_{\omega^{\alpha_1}\#\omega^{\alpha_2}} \Gamma$. We can conclude because since $\alpha_1, \alpha_2 < \alpha$, we have that $\omega^{\alpha_1}\#\omega^{\alpha_2} \leq \omega^\alpha$.

□

Recall that $\omega_k(\alpha)$ for α an ordinal and k a natural number is defined inductively by $\omega_0(\alpha) = \alpha$ and $\omega_{k+1}(\alpha) = \omega^{\omega_k(\alpha)}$.

Theorem 4. *If $\vdash^k_\alpha \Gamma$ with $\alpha < \varepsilon_0$ and k a natural number, then $\vdash^0_{\omega_k(\alpha)} \Gamma$.*

Proof. This is provable by induction on k, using Lemma 42 and the elementary fact that $\alpha < \varepsilon_0$ imply $\omega^\alpha < \varepsilon_0$. □

Note that the only properties of ε_0 we used are :

- It is well-founded. This can not be proven in PA.
- If $\alpha < \varepsilon_0$ then $\omega^\alpha < \varepsilon_0$.
- If $\alpha, \alpha' < \varepsilon_0$ then $\alpha \# \alpha' < \varepsilon_0$.

B.3.2 The translation of expansion proofs to strategies

In this section we are going to prove the link between expansion proofs and strategies. We will mix finitary and infinitary formulas.

We begin by the copy-cat lemma.

Lemma 43. *For any infinitary formula A of depth α and any Γ, we have $\vdash^0_{2\times\alpha} \Gamma, A, \overline{A}$.*

Proof. We proceed by induction on the depth of A.

- If depth of A is 0, then $A = \top$ or $\bot$ and this is obvious.
- Otherwise $A = \bigwedge_k A_k$. By inductive hypothesis for all k we have $\vdash^0_{2\times\alpha_k} \Gamma, A_k, \overline{A_k}$ with α_k the depth of A_k. So $\vdash^0_{2\times\alpha_k} \Gamma, A_k, \bigvee_k \overline{A_k}, \overline{A_k}$ by weakening, by Lemma 3 we have $\vdash^0_{2\times\alpha_k+1} \Gamma, A_k, \bigvee_k \overline{A_k}$, so by Lemma 4 we have that $\vdash^0_{\sup^*_k 2\times\alpha_k+1} \Gamma, \bigwedge_k A_k, \bigvee \overline{A_k}$. But $\sup^*_k 2\times\alpha_k + 1 = \sup_k 2\times\alpha_k + 2 = \sup_k 2\times(\alpha_k+1) \leq 2\times\alpha$.

□

Lemma 44. *If there is an expansion proof with shallow sequent Γ and free variables $\alpha_1, .., \alpha_n$, then for any $k_1, \dots, k_n$ natural numbers, we have that $\vdash^l_{\omega^2} (\Gamma[\alpha_1/k_1, \dots \alpha_n/k_n])^\infty$ with l a natural number.*

Proof. By Theorem 1, it is enough to prove that if there is a proof of Γ in PA, then $\vdash^l_{\omega^2}$ $(\Gamma[\alpha_1/k_1, \dots \alpha_n/k_n])^\infty$ with l a natural number.

For notational convenience we denote the family of $[\alpha_1/k_1, \dots \alpha_n/k_n]$ for $k_1, \dots, k_n$ natural numbers by $(\sigma_i)_{i\in\omega}$. We write $\vdash^k_{<\alpha} \Gamma$ to mean that $\vdash^k_{\alpha'} \Gamma$ for some $\alpha' < \alpha$.

Assume all formulas appearing in the given proof are of depth $< l$ with l a natural number. It will be enough to prove inductively (on the subproof of the given proof) that if there is a proof in PA of Γ then for all natural numbers i we have $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty$, where k is the number of nested $\forall$ rule in the subproof.

- If we are dealing with an axiom $\overline{A}, A$ we use a copycat strategy from Lemma 43, which is of ordinal $< \omega$ because the depths of the formulas in Γ^∞ are finite.
- If we are dealing with an axiom from Q^-, it is obvious that $(\Gamma\sigma_i)^\infty$ contains $\top$ for any i.
- If we are dealing with a cut rule, let i be a natural number. By induction hypothesis $\vdash^l_{<\omega\times k}$ $(\Gamma\sigma_i)^\infty, (A\sigma_i)^\infty$ and $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (\overline{A}\sigma_i)^\infty$. But notice that $(\overline{A})^\infty$ is $\overline{A^\infty}$. Moreover the depth of A is strictly smaller than l by definition of l. Therefore by Lemma 5 we have that $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty$.

- If we are dealing with an $\wedge$ rule, let i be a natural number. Then by induction hypothesis $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (A\sigma_i)^\infty$ and $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (B\sigma_i)^\infty$. Using Lemma 4 we know that $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (A\sigma_i \wedge B\sigma_i)^\infty$.

- If we are dealing with a $\vee$ rule, let i be a natural number. By induction hypothesis $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (A\sigma_i)^\infty, (B\sigma_i)^\infty$. Using weakening we have $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (A\sigma_i \vee B\sigma_i)^\infty, (A\sigma_i)^\infty, (B\sigma_i)^\infty$. By Lemma 3 (used twice) we have that $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (A\sigma_i \vee B\sigma_i)^\infty$

- If we are dealing with a $\forall$ rule,
  $$\frac{\Gamma, A(\alpha_{n+1})}{\Gamma, \forall x.A(x)}$$
  let i be a natural number. By induction hypothesis, for all natural number k_{n+1} we have $\vdash_{<\omega\times k} (\Gamma\sigma_i)^\infty, (A(k_{n+1})\sigma_i)^\infty$. Indeed there is one more free variable α_{n+1} in the given proof (recall x was substituted by α_{n+1}, recall $A(k_{n+1})$ is $A[x/k_{n+1}]$). But α_{n+1} is not occurring in Γ by the side condition of this rule, so we have indeed substituted correctly. From this using Lemma 4 we know that $\vdash^l_{\omega\times k} (\Gamma\sigma_i)^\infty, (\forall x.A(x)\sigma_i)^\infty$. But since k is the number of nested $\forall$ rule this is okay.

- If we are dealing with a $\exists$ rule
  $$\frac{\Gamma, \exists x.A(x), A(t)}{\Gamma, \exists x.A(x)}$$
  note that we may assume that all the variables in t occurs in $\Gamma, \exists x.A(x)$ by elementary sequent calculus manipulations. Let i be a natural number. By induction hypothesis $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (\exists x.A(x)\sigma_i)^\infty, (A(t)\sigma_i)^\infty$. But $t\sigma_i$ is a natural number, and therefore $(A(t)\sigma_i)^\infty$ occurs in the $\bigvee_k (A(k)\sigma_i)^\infty = (\exists x.A(x)\sigma_i)^\infty$. Therefore using Lemma 3, we have the desired result $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (\exists x.A(x)\sigma_i)^\infty$.

- If we are dealing with the Ind rule, let i be a natural number. By induction hypothesis $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (A(0)\sigma_i)^\infty$ and $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (\overline{A}(t)\sigma_i)^\infty$ and $\vdash^l_{<\omega\times k} (\Gamma\sigma_i)^\infty, (\forall x.\overline{A}(x) \vee A(x)\sigma_i)^\infty$. In this case $t\sigma_i$ is natural number, say n. To simplify notations we omit the σ_i and we admit Γ and $A(t)$ closed with $t = n$. We will show by backward induction on j that
  $$\vdash^l_{<\omega\times k} \Gamma^\infty, \overline{A}(0)^\infty, \ldots, \overline{A}(j)^\infty, (A(j) \wedge \overline{A}(j+1))^\infty, \ldots, (A(n-1) \wedge \overline{A}(n))^\infty$$
  for $0 \le j \le n$.

  We begin by $j = n$. We need to prove $\vdash^l_{<\omega\times k} \Gamma^\infty, \overline{A}(0)^\infty, \ldots, \overline{A}(n)^\infty$. But by inductive hypothesis $\vdash^l_{<\omega\times k} \Gamma^\infty, \overline{A}(n)^\infty$, so we can conclude using a weakening.

  Assume it is true for $j > 0$, show it is true for $j-1$. We want
  $$\vdash^l_{<\omega\times k} \Gamma^\infty, \overline{A}(0)^\infty, \ldots, \overline{A}(j-1)^\infty, (A(j-1) \wedge \overline{A}(j))^\infty, \ldots, (A(n-1) \wedge \overline{A}(n))^\infty$$
  We use Lemma 4 on $(A(j-1) \wedge \overline{A}(j))^\infty$. So we just have to show two things.

  Firstly :
  $$\vdash^l_{<\omega\times k} \Gamma^\infty, \overline{A}(0)^\infty, \ldots, \overline{A}(j-1)^\infty, A(j-1)^\infty, \ldots, (A(n-1) \wedge \overline{A}(n))^\infty$$
  If $j > 1$ we use Lemma 43 (recall that depth of A is finite and $k \ge 1$). If $j = 0$ we use the inductive hypothesis $\vdash^l_{<\omega\times k} \Gamma^\infty, A(0)^\infty$.

  Secondly :
  $$\vdash^l_{<\omega\times k} \Gamma^\infty, \overline{A}(0)^\infty, \ldots, \overline{A}(j-1)^\infty, \overline{A}(j)^\infty, \ldots, (A(n-1) \wedge \overline{A}(n))^\infty$$
  This is the inductive hypothesis.

So we know that

$$\vdash^l_{<\omega\times k} \Gamma^\infty, (A(0) \wedge \overline{A}(1))^\infty, \dots, (A(n-1) \wedge \overline{A}(n))^\infty$$

From this we can conclude using weakening on $(\exists x.A(x) \wedge \overline{A}(x+1))^\infty$ and Lemma 3 that $\vdash^l_{<\omega\times k} \Gamma^\infty, (\exists x.A(x) \wedge \overline{A}(s(x)))^\infty$. Since the depth of $\exists x.A(x) \wedge \overline{A}(x+1)$ is $< l$ by definition of l, we can conclude using the inductive hypothesis $\vdash^l_{<\omega\times k} \Gamma^\infty, (\forall x.\overline{A}(x) \vee A(x+1))^\infty$ and Lemma 5.

□

Note the last case of induction is represented graphically in Section 3.3.1.

Lemma 6. *Let Γ be a closed arithmetical sequent. If there exists an expansion proof with shallow sequent Γ, then $\vdash^l_{\omega^2} \Gamma^\infty$ with l finite.*

Proof. This is an immediate consequence of Lemma 44. □

B.4 The full cut-elimination result

Here we present the cut-elimination theorem in full : we can eliminate cuts from any strategy. We again closely follow [20].

This theorem is interesting in its own right because it establishes a nice game-theoretic interpretation of strategies, even with cuts. But it uses an induction up to ω_1, because it is the only bound a priori on the ordinal of a given strategy.

First we define the Veblen hierarchy [23], which will allow us to measure the increase of the ordinal of a strategy during the cut-elimination procedure.

Definition 41. *We define inductively on α :*

- $\phi_0(\beta) = \omega^\beta$.
- *$\phi_\alpha(\beta)$ is the β-th fixpoint of the ϕ_γ for $\gamma < \alpha$.*

We recall without proof a basic fact about the Veblen hierarchy.

Lemma 45. *For each γ, the function ϕ_γ is normal, meaning that :*

- *It is strictly increasing.*
- *For any family of ordinals $(\alpha_i)_{i\in I}$, we have $\sup_i(\phi_\gamma(\alpha_i)) = \phi_\gamma(\sup_i(\alpha_i))$.*

Lemma 46. *If for A such that $d(A) \leq \beta$ we have $\vdash^\beta_\alpha \Gamma, A$ and $\vdash^\beta_{\alpha'} \Gamma, \overline{A}$, then $\vdash^\beta_{\alpha\#\alpha'} \Gamma$.*

Proof. The proof is exactly the same as the proof of Lemma 41. □

Theorem 7. *If $\vdash^{\beta+\omega^\gamma}_\alpha \Gamma$ then $\vdash^\beta_{\phi_\gamma(\alpha)} \Gamma$.*

Proof. By induction we assume the result is true for $\alpha' < \alpha$ and γ, and true for any α' and $\gamma' < \gamma$.

We reason by case distinction on the reductions of the strategies.

- Assume the strategy is in normal form, then it is easy to obtain a cut-free strategy in normal form.
- Otherwise assume the strategy has a non-cut reduction.

  This reduction is on $\exists$, then $\Gamma = \Gamma', \bigvee_i A_i$ and $\vdash^{\beta+\omega^\gamma}_{\alpha'} \Gamma', \bigvee_i A_i, A_k$ for $\alpha' < \alpha$. By induction hypothesis $\vdash^\beta_{\phi_\gamma(\alpha')}, \Gamma', \bigvee_i A_i, A_k$. But $\phi_\gamma(\alpha') < \phi_\gamma(\alpha)$ so we can conclude by Lemma 3.

  This reduction is on $\forall$, then $\Gamma = \Gamma', \bigwedge_i A_i$ and $\vdash^{\beta+\omega^\gamma}_{\alpha_i} \Gamma', A_i$ for all i, with $\alpha_i < \alpha$. By induction hypothesis $\vdash^\beta_{\phi_\gamma(\alpha_i)} \Gamma', A_i$. Then by Lemma 4 we have $\vdash^\beta_{\sup^*_i \phi_\gamma(\alpha_i)} \Gamma$. But $\sup^*_i \phi_\gamma(\alpha_i) \leq \phi_\gamma(\alpha)$ so we can conclude.

- Otherwise the proof has a cut-reduction. This means we have $\vdash^{\beta+\omega^\gamma}_{\alpha_1} \Gamma, A$ and $\vdash^{\beta+\omega^\gamma}_{\alpha_2} \Gamma, \overline{A}$ for some A with $d(A) < \beta + \omega^\gamma$ and $\alpha_1, \alpha_2 < \alpha$. By induction hypothesis $\vdash^{\beta}_{\phi_\gamma(\alpha_1)} \Gamma, A$ and $\vdash^{\beta}_{\phi_\gamma(\alpha_2)} \Gamma, \overline{A}$. Here we proceed by case on γ :

  If $\gamma = 0$, then $d(A) < \beta + 1$ so $d(A) \leq \beta$. By applying Lemma 46 we know that $\vdash^{\beta}_{\phi_0(\alpha_1)\#\phi_0(\alpha_2)} \Gamma$. All we need to conclude is $\phi_0(\alpha_1)\#\phi_0(\alpha_2) = \omega^{\alpha_1}\#\omega^{\alpha_2} \leq \omega^\alpha = \phi_0(\alpha)$, which is true since $\alpha_1, \alpha_2 < \alpha$.

  If $\gamma \neq 0$, then $d(A) < \beta + \omega^{\gamma'} \times k$ for k a natural number and $\gamma' < \gamma$. Let us denote $\delta = \max(\phi_\gamma(\alpha_1), \phi_\gamma(\alpha_2)) + 1$. Then using a cut on A we have $\vdash^{\beta+\omega^{\gamma'}\times k}_{\delta} \Gamma$, so by induction $\vdash^{\beta}_{\phi^k_{\gamma'}(\delta)} \Gamma$. But notice $\phi^k_{\gamma'}(\delta) \leq \phi^k_{\gamma'}(\phi_\gamma(\alpha)) = \phi_\gamma(\alpha)$ so we are done.

$\square$